The Mathieu Groups

(Simple Sporadic Symmetries)

Scott Harper
(University of St Andrews)

Tomorrow's Mathematicians Today
21st February 2015

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Symmetry

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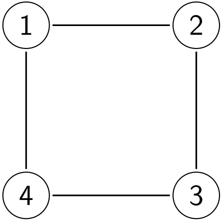

Symmetry group: D_4

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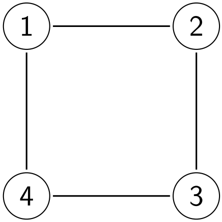
A symmetry is a structure preserving permutation of the underlying set.



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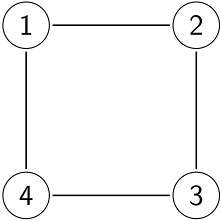

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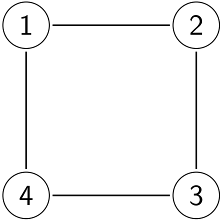

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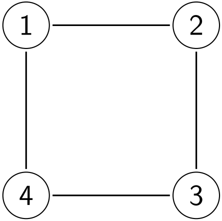

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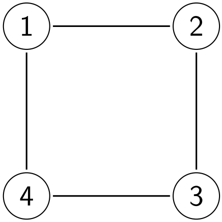
The stabiliser of a point in a group G is the subgroup of G which fixes x.



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The alternating groups of degree at least five are simple.

Alternating Groups: Shuffles

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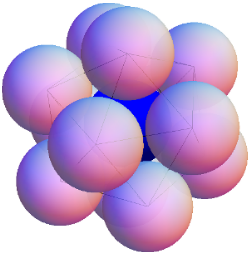
Alternating group A_{12}.

Alternating Groups: Kissing Spheres



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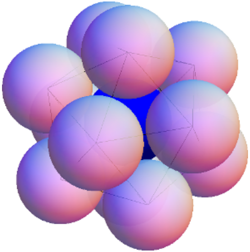

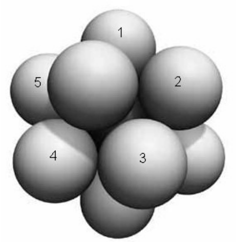

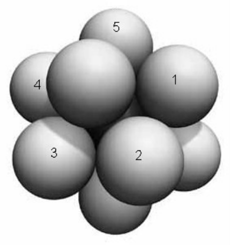

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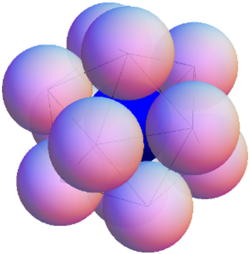

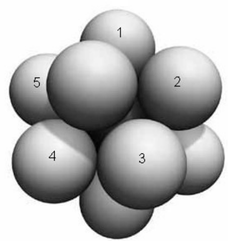

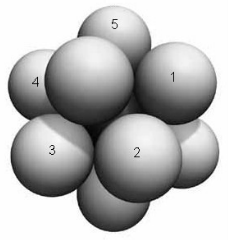

Twisting group:
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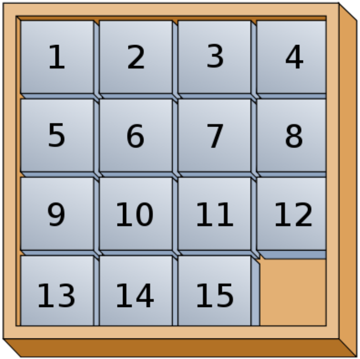
Alternating Groups: A Puzzle



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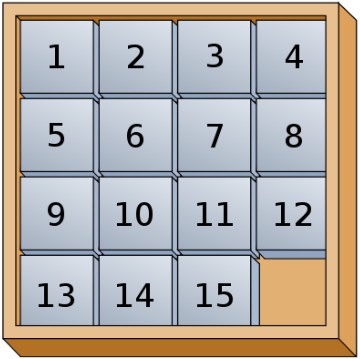
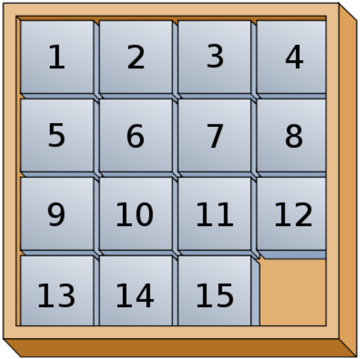


A transposition of the contents of space p and space q, written $[p, q]$, is called an elementary move. This is valid if one of p or q is empty.



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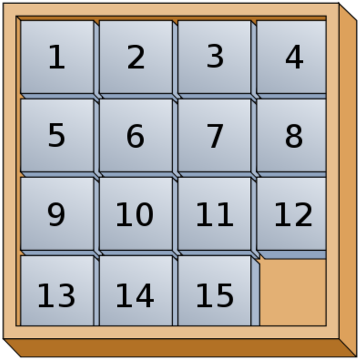

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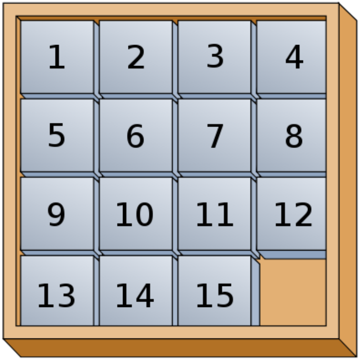

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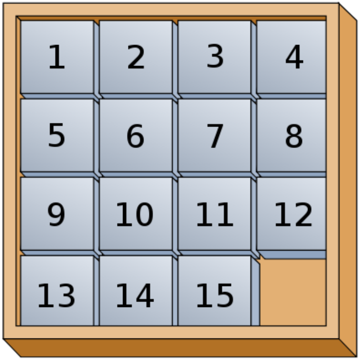

Associated group:
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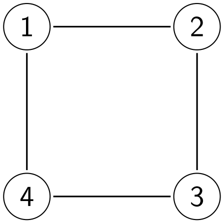

Transitivity

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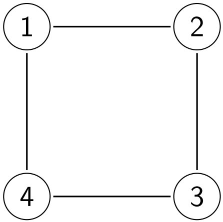
Let G be a group acting on a set X. We say G acts



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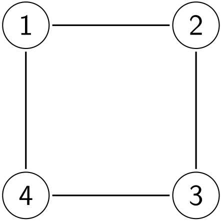

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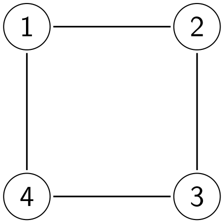

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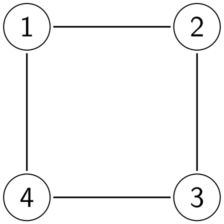
- sharply k-transitively if the above g is the unique such element.



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Theorem (The Classification of Finite Simple Groups)

Every finite simple group is isomorphic to one of the following groups:

- *a cyclic group of prime order;*
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Let G be a group acting k-transitively on a set. If G is not a symmetric or alternating group then $k \leq 5$. Moreover,

- *if $k = 5$ then G is the Mathieu group M_{12} or M_{24};*
- *if $k = 4$ then G is the Mathieu group M_{11} or M_{23}.*

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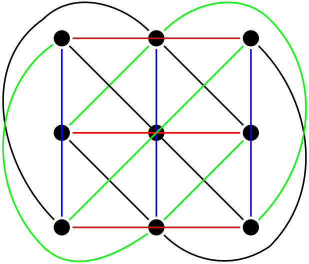
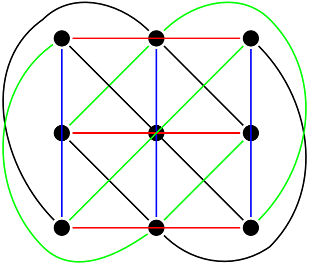
Note: M_{22} acts 3-transitively.

Witt Geometries



The affine plane $AG_2(3)$



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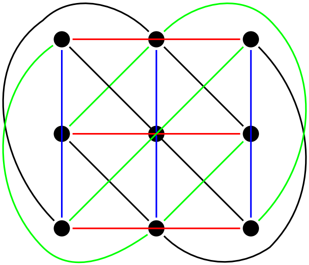
An $S(t,k,v)$ Steiner system is a set X of v points and a set of k-element subsets of X such that any t points lie in a unique such subset.



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Example

The affine plane $AG_2(3)$ is an $S(2, 3, 9)$ Steiner system.

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- It is 2-transitive.
- Index 6 subgroup is sharply 2-transitive.

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Suppose that

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- G_x *acts* $(k-1)$*-transitively on* $X \setminus \{x\}$*, for some* $x \in X$.

Then G *acts* k*-transitively on* X.

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Then G is simple.

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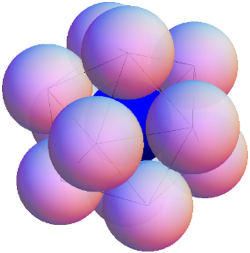
Mathieu group M_{12}.



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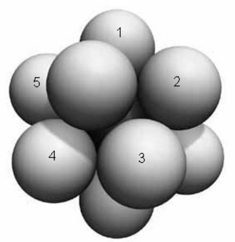

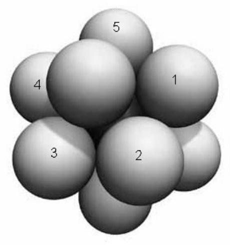

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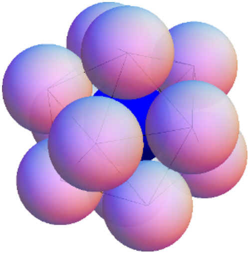

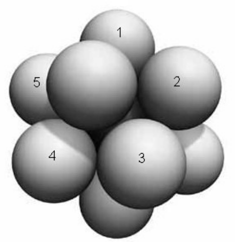

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Twist-untwist group:
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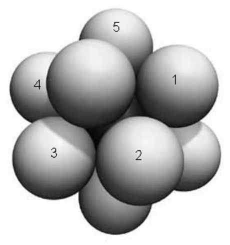

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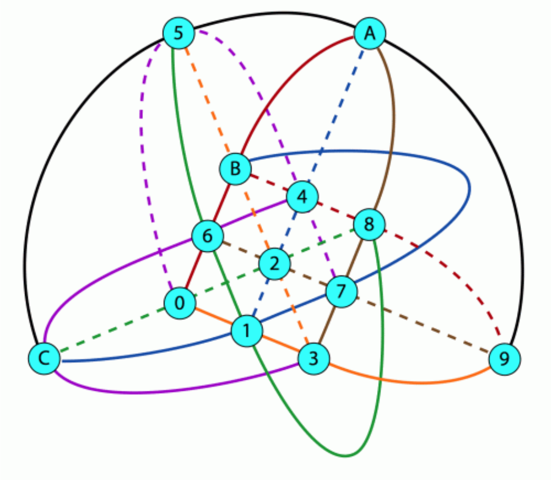
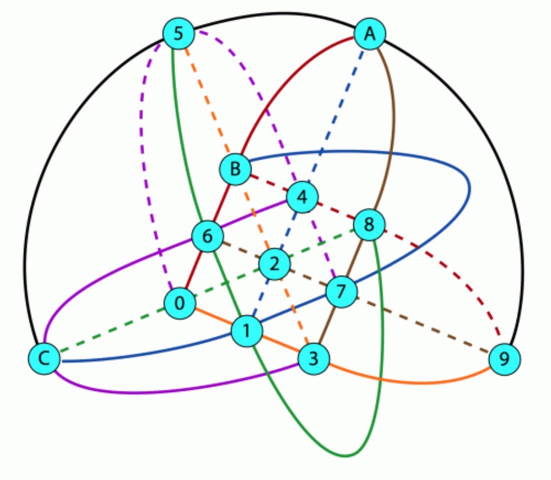


(Image by Bob Harris)



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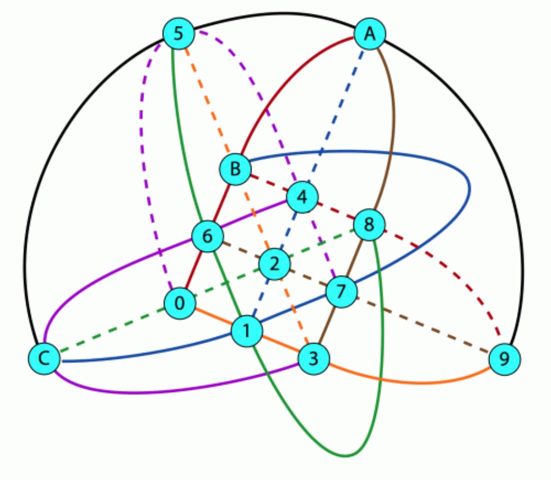
Elementary moves:

- Move a tile from p to the empty space
- Look at the unique line which p and empty space define.
- Swap the other two tiles on this line.



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Associated group:
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Any questions?