Sporadic Symmetry

(The Remarkable Behaviour of the Mathieu Groups)

Scott Harper

MT5999 Presentation

17th April 2015

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It is a remarkable fact that ...

From *Sphere Packings, Lattices and Groups* by Conway and Sloane,

> "At one point while working on this book we even considered adopting a special abbreviation for 'It is a remarkable fact that' since this phrase seemed to occur so often. But in fact we have tried to avoid such phrases and to maintain a scholarly decorum of language."

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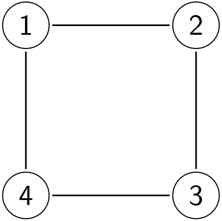
Theorem

The symmetric group S_n has a non-trivial outer automorphism if and only if $n = 6$.

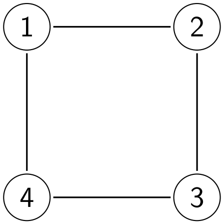
Transitivity



Transitivity

Definition

Let G be a group acting on a set X. We say G acts



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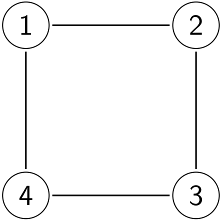
Let G be a group acting on a set X. We say G acts

- transitively if for all $x, y \in X$ there exists $g \in G$ such that $xg = y$;



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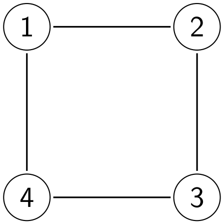
- k-transitively if for all sequences of distinct points $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in X^k$ there exists $g \in G$ such that $x_i g = y_i$;



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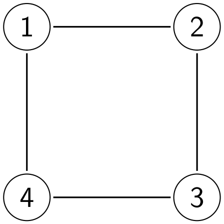
- sharply k-transitively if the above g is the unique such element.



Simplicity

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Theorem (The Classification of Finite Simple Groups)

Every finite simple group is isomorphic to one of the following groups:

- *a cyclic group of prime order;*
- *an alternating group of degree at least 5;*
- *a simple group in one of the 16 families of groups of Lie type;*
- *one of the 26 sporadic simple groups.*

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Let G be a group acting k-transitively on a set. If G is not a symmetric or alternating group then $k \leq 5$. Moreover,

- *if $k = 5$ then G is the Mathieu group M_{12} or M_{24};*
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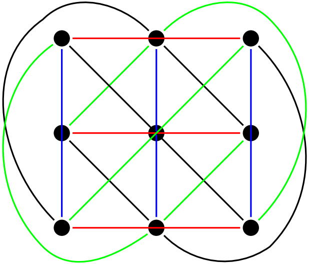
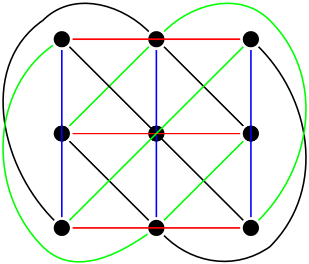
Note: M_{22} acts 3-transitively.

Steiner Systems



The affine plane $AG_2(3)$

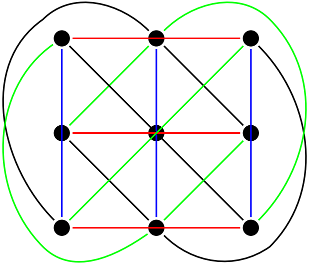


The affine plane $AG_2(3)$

Definition

An $S(t, k, v)$ Steiner system is a set X of v points and a set of k-element subsets of X such that any t points lie in a unique such subset.

Steiner Systems



The affine plane $AG_2(3)$

Definition

An $S(t, k, v)$ Steiner system is a set X of v points and a set of k-element subsets of X such that any t points lie in a unique such subset.

Example

The affine plane $AG_2(3)$ is an $S(2, 3, 9)$ Steiner system.

Witt Geometries and Mathieu Groups

$S(2, 3, 9)$ $\quad AG_2(3)$

$S(3,4,10)$

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Witt Geometries and Mathieu Groups

$S(4,5,11)$

$S(3,4,10)$

$S(2,3,9)$ $\quad AG_2(3)$

Witt Geometries and Mathieu Groups

$S(5, 6, 12)$

$S(4, 5, 11)$

$S(3, 4, 10)$

$S(2, 3, 9)$ $\quad AG_2(3)$

Witt Geometries and Mathieu Groups

$S(5,6,12)$	W_{12}
$S(4,5,11)$	W_{11}
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$S(2,3,9)$	$AG_2(3)$

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$$\mathrm{Aut}(AG_2(3)) \\ \| \\ AGL_2(3)$$

Witt Geometries and Mathieu Groups

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$$\begin{array}{ccc} & & \mathrm{Aut}(AG_2(3)) \\ & & \| \\ {\color{red}H} & \leq & AGL_2(3) \end{array}$$

$S(5,6,12)$	W_{12}	M_{12}	$:=$	$\mathrm{Aut}(W_{12})$
$S(4,5,11)$	W_{11}			$\mathrm{Aut}(W_{11})$
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Theorem

Suppose that

- *G acts transitively on X*
- *G_x acts k-transitively on $X \setminus \{x\}$, for some $x \in X$.*

Then G acts $(k+1)$-transitively on X.

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Suppose that

- *G acts k-transitively on X for $k \geq 3$ and $|X|$ not equal to 3 or 2^n*
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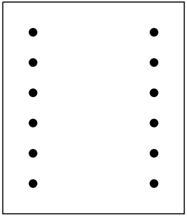

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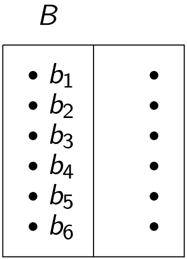

M_{12} acting on W_{12}

B	C
• b_1	c_1 •
• b_2	c_2 •
• b_3	c_3 •
• b_4	c_4 •
• b_5	c_5 •
• b_6	c_6 •

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G

M_{12} acting on W_{12}

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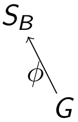
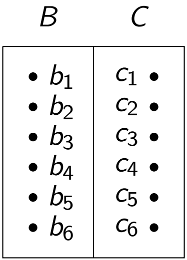
$\phi : g \mapsto g\,|_B$





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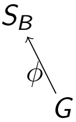
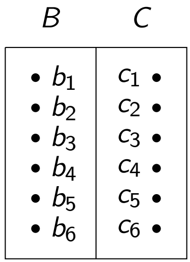


The map ϕ is a homomorphism,



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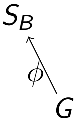
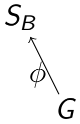


The map ϕ is a homomorphism, an injection

M_{12} acting on W_{12}

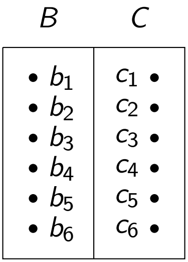
B	C
• b_1	c_1 •
• b_2	c_2 •
• b_3	c_3 •
• b_4	c_4 •
• b_5	c_5 •
• b_6	c_6 •

$\phi : g \mapsto g\,|_B$

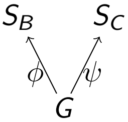


The map ϕ is a homomorphism, an injection and a surjection.

M_{12} acting on W_{12}



$\phi : g \mapsto g\,|_B$



$\psi : g \mapsto g\,|_C$

The map ϕ is a homomorphism, an injection and a surjection.

M_{12} acting on W_{12}

B	C
• b_1	c_1 •
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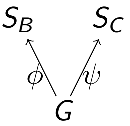
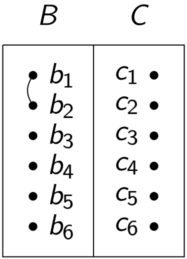


$\psi : g \mapsto g\,|_C$

The map ϕ is a homomorphism, an injection and a surjection.

Fact: $\phi^{-1}\psi : S_B \to S_C$ is an isomorphism.

M_{12} acting on W_{12}



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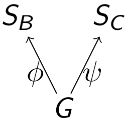

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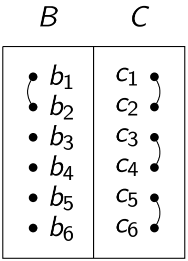

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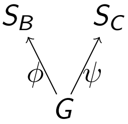

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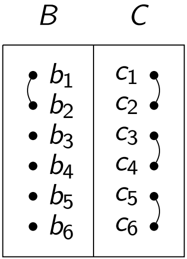

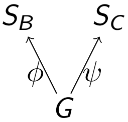

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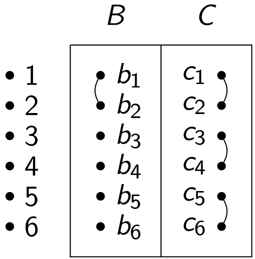

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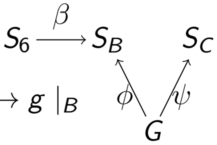

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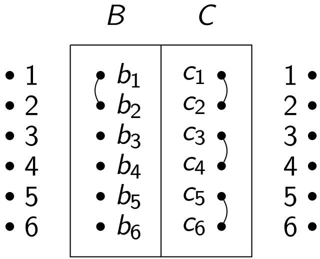

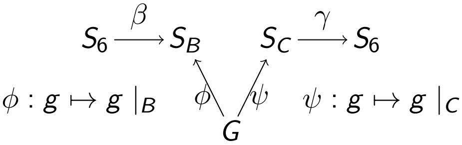

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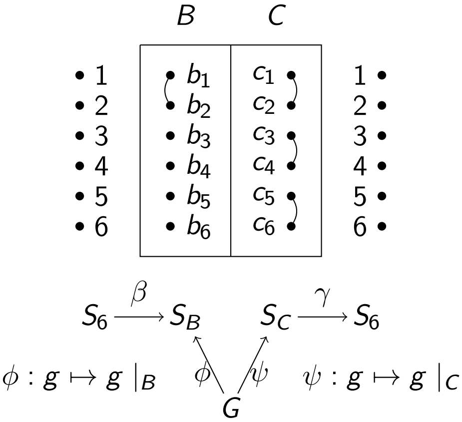

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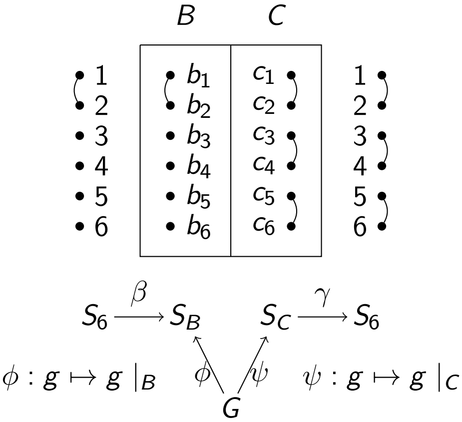

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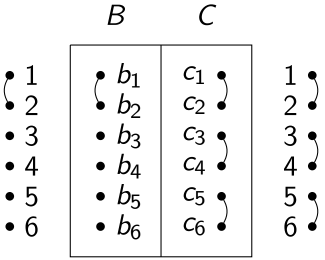

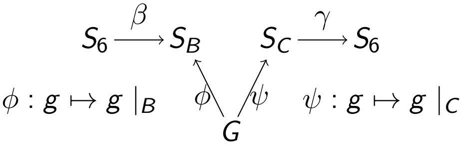

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So $\beta\phi^{-1}\psi\gamma$ is a non-trivial outer automorphism of S_6.

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(William Burnside)

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Any questions?