Throwing Darts at Fractals

Scott Harper
(On work with D. Allen, H. Edwards, L. Olsen)

Pure Postgraduate Seminar

13th November 2015

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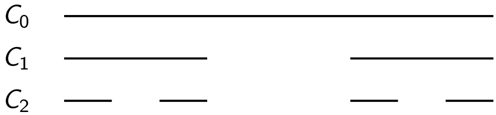

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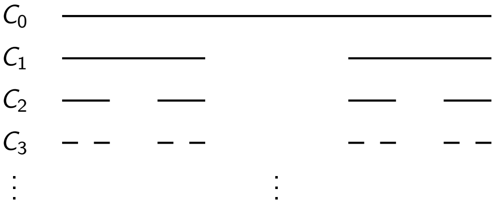

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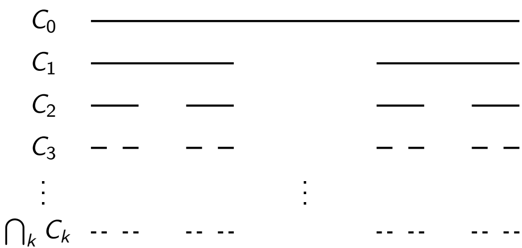

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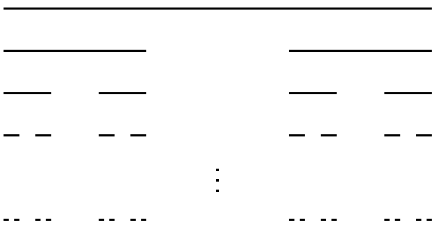

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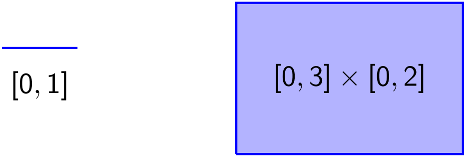
Hausdorff Measure and Dimension

$[0, 1]$


[0, 1]
[0, 3] × [0, 2]

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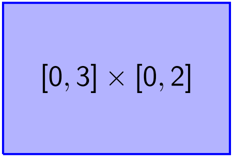
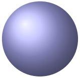
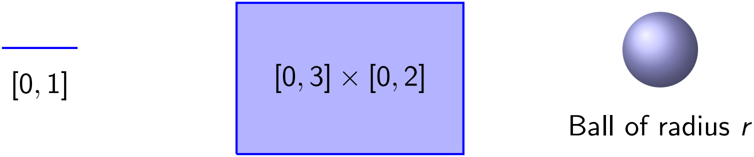




Ball of radius r

Hausdorff Measure and Dimension



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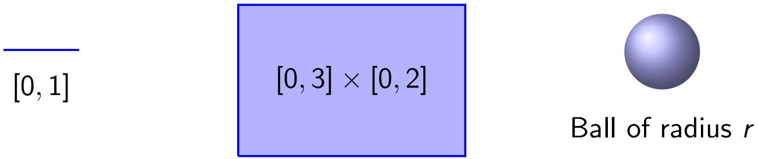

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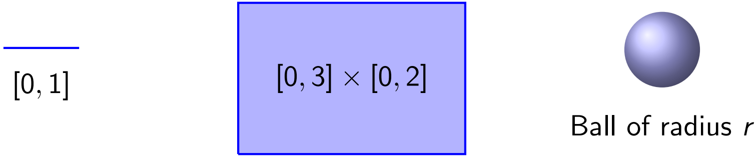

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We call d the Hausdorff dimension of A.

Average Distance

Recall: The limiting average distance of C is

$$A_L := \lim_k \frac{1}{\mathcal{L}(C_k)^2} \int_{C_k^2} |x - y| d(x, y).$$

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Proposition

For the middle-third Cantor set $A_L = A_H = \frac{2}{5}$.

Sketch of Proof

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For each $k \in \mathbb{N}$ let $L_k = [0, \frac{1}{3}] \cap C_k$ and let $R_k = [\frac{2}{3}, 1] \cap C_k$.

$$\int_{C_{k+1}^2} |x - y| d(x, y) = 2 \int_{L_{k+1}^2} |x - y| d(x, y) + 2 \int_{L_{k+1}} \int_{R_{k+1}} |x - y| dx dy$$

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We obtain the recurrence relation

$$A(C_{k+1}) = \frac{1}{6}A(C_k) + \frac{1}{3}.$$

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Solving as $k \to \infty$ gives

$$A_L(C) = \frac{1}{1 - \frac{1}{6}} \frac{1}{3} = \frac{2}{5}.$$

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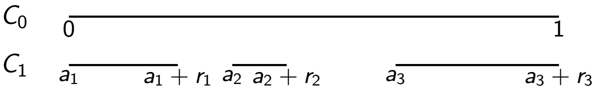

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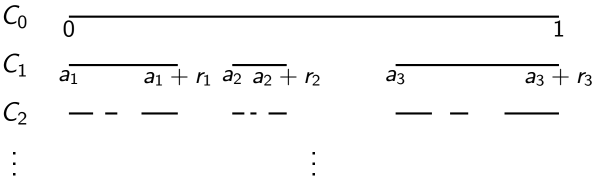

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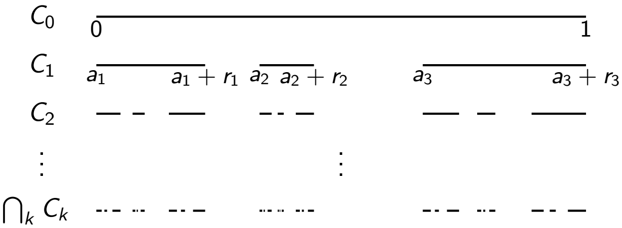

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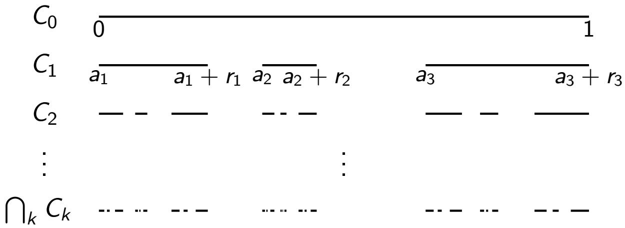

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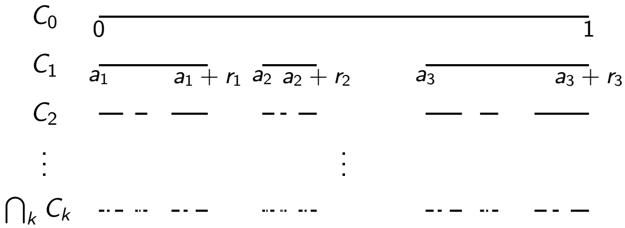

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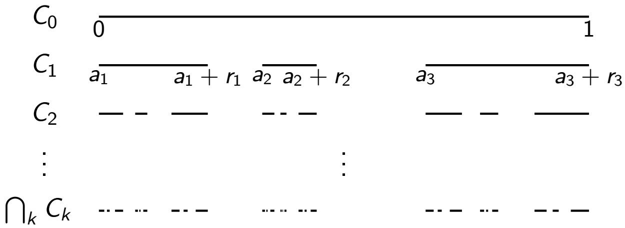

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The corresponding average distance of C_k is

$$A_{\mathrm{geo},k}(\mathbf{p},\mathbf{q}) = \sum_{|\mathbf{i}|=|\mathbf{j}|=k} \frac{\mu(I_{\mathbf{i}})\nu(I_{\mathbf{j}})}{r_{\mathbf{i}} r_{\mathbf{j}}} \int_{I_{\mathbf{i}} \times I_{\mathbf{j}}} |x-y| \, d(x,y)$$

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We show that $A_{\mathrm{geo}}(\mathbf{p},\mathbf{q}) = A(\mu_{\mathbf{p}},\mu_{\mathbf{q}})$.

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If $\mathbf{p} = \mathbf{q} = \left(\frac{r_1}{S}, \frac{r_2}{S}, \ldots, \frac{r_N}{S}\right)$ where $S = \sum_i r_i$ then $A(\mu_\mathbf{p}, \mu_\mathbf{q}) = A_L$.

Theorem

For an IFS C satisfying the OSC with contraction ratios $r_1, r_2, \ldots, r_N$ there is a unique real number s such that $\sum_i r_i^s = 1$. Moreover, $s = \dim_H(C)$.

If $\mathbf{p} = \mathbf{q} = (r_1^s, r_2^s, \ldots, r_N^s)$ where $s = \dim_H(C)$ then $A(\mu_\mathbf{p}, \mu_\mathbf{q}) = A_H$.

The Average Distance Theorem

For $\pi = (\pi_1, \pi_2, \dots, \pi_N)$ write $B_\pi = \frac{\sum_i \pi_i a_i}{1 - \sum_i \pi_i r_i}$. Write $s_{i,j}$ for $\mathrm{sign}(i - j)$.

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Let C be defined as above in terms of the sequences $(a_i)_i$ and $(r_i)_i$ and let $\mathbf{p} = (p_1, p_2, \ldots, p_N)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ be probability vectors. Then

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The result follows. ∎

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Corollary (Leary et al.)

$A_L = \frac{1}{S^2 - \sum_i r_i^3} \left(\sum_{i,j} r_i r_j |a_i - a_j| + 2 \frac{\sum_i r_i a_i}{S - \sum_i r_i^2} \sum_{i,j} s_{i,j} r_i^2 r_j \right)$ *where* $S = \sum_i r_i$.

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Corollary

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Example

Consider C where $N = 2$, $a_1 = 0$, $a_2 = \frac{1}{2}$, $r_1 = \frac{1}{4}$ and $r_2 = \frac{1}{2}$. Then $A_L = 8/21 \simeq 0.381$ and $A_H = \frac{12}{5(4+\sqrt{5})} \simeq 0.385$.

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Expression for periodic function: $\Pi(u) = \frac{1}{-\log r} \sum_{n \in \mathbb{Z}} Z(s_n) e^{2\pi i \frac{\log u}{\log r}}$.

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We want to show that for some periodic function Π and a sequence $(\varepsilon_n)_n$ such that $\varepsilon_n \to 0$ we have

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We have

$$\left|M_n - n^{-\Delta}\Pi(n)\right| \leq |M_n - L(n)| + \left|L(n) - n^{-\Delta}\Pi(n)\right|.$$

where $L : \mathbb{C} \to \mathbb{C}$ is defined as

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Theorem (Jacquet-Szpankowksi)

Let $(t_n)_n$ be a sequence of bounded positive numbers and define $f : \mathbb{C} \to \mathbb{C}$ by $f(s) = \sum_n \frac{t_n}{n!} s^n e^{-s}$.

Suppose that there exist $\theta < \frac{\pi}{2}$ and $R > 0$ such that for $|s| > R$

1. $s \in \mathcal{S}_\theta \implies |f(s)| \leq A\frac{1}{|s|^D}$ *for some $A, D > 0$;*
2. $s \notin \mathcal{S}_\theta \implies |f(s)e^s| \leq Be^{\delta|s|}$ *for some $B, 0 < \delta < 1$.*

Then there is a constant C such that $|t_n - f(n)| \leq C\frac{1}{n^{D+1}}$ for all $n \in \mathbb{N}$.

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There is a constant C such that $|M_n - L(n)| \leq C_1\frac{1}{n^{\Delta+1}} = n^{-\Delta}\frac{C_1}{n}$.

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Definition (The Mellin Transform)

Let $a, b \in [-\infty, \infty]$ with $a < b$ and let $f : (0, \infty) \to \mathbb{R}$ be piecewise continuous on all compact subintervals of $(0, \infty)$ with $f(x_0) = \frac{lim_{x \searrow x_0} f(x) + lim_{x \nearrow x_0} f(x)}{2}$ at all discontinuity points x_0 of f.

The Mellin transform of f $\mathrm{M}f : \{s \in \mathbb{C} \mid a < \mathrm{Re}(s) < b\} \to \mathbb{C}$ is defined as

$$\mathrm{M}f(s) = \int_0^\infty x^{s-1} f(x) dx,$$

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Theorem (The Mellin Inversion Theorem)

Consider the same setting as above. For $a < c < b$ and $x > 0$ we have

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} (Mf)(s) ds.$$

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$$2\pi|L(u) - u^{-\Delta}\Pi(u)| = \left|\int_{c-i\infty}^{c+i\infty} f(s)ds - 2\pi i \sum_{n\in\mathbb{Z}} \mathrm{r}(s_n)\right|$$

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\begin{aligned}
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&\leq \left|\int_{c-i\infty}^{c+i\infty} f(s)ds - \int_{c-ia_N}^{c+ia_N} f(s)ds\right| + \left|\int_{c-ia_N}^{c+ia_N} f(s)ds - 2\pi i \sum_{n\in\mathbb{Z}} \mathrm{r}(s_n)\right|
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Let $\varepsilon > 0$. Choose $N \geq N_1$

$$2\pi|L(u) - u^{-\Delta}\Pi(u)| = \left|\int_{c-i\infty}^{c+i\infty} f(s)ds - 2\pi i \sum_{n\in\mathbb{Z}} \mathrm{r}(s_n)\right|$$

$$\leq \frac{\varepsilon}{4} + \left|\int_{c-ia_N}^{c+ia_N} f(s)ds - 2\pi i \sum_{n\in\mathbb{Z}} \mathrm{r}(s_n)\right|$$

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&\leq \frac{\varepsilon}{4} + \left|\int_{c-ia_N}^{c+ia_N} f(s)ds - 2\pi i \sum_{|n|\leq N} \mathrm{r}(s_n)\right| + 2\pi i \left|\sum_{|n|\leq N} \mathrm{r}(s_n) - \sum_{n\in\mathbb{Z}} \mathrm{r}(s_n)\right|
\end{aligned}
$$

Let $\varepsilon > 0$. Choose $N \geq N_1, N_2$

$$\begin{aligned}
2\pi|L(u) - u^{-\Delta}\Pi(u)| &= \left| \int_{c-i\infty}^{c+i\infty} f(s)ds - 2\pi i \sum_{n\in\mathbb{Z}} \mathrm{r}(s_n) \right| \\
&\leq \frac{\varepsilon}{4} + \left| \int_{c-ia_N}^{c+ia_N} f(s)ds - 2\pi i \sum_{n\in\mathbb{Z}} \mathrm{r}(s_n) \right| \\
&\leq \frac{\varepsilon}{4} + \left| \int_{c-N}^{c+ia_N} f(s)ds - 2\pi i \sum_{|n|\leq N} \mathrm{r}(s_n) \right| + \frac{\varepsilon}{4}
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&= \frac{\varepsilon}{4} + \left|\int_{c-ia_N}^{c+ia_N} f(s)ds - \int_{\Gamma_N} f(s)ds\right| + \frac{\varepsilon}{4}
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&= \frac{\varepsilon}{4} + \left|\int_{c-ia_N}^{c+ia_N} f(s)ds - \int_{\Gamma_N} f(s)ds\right| + \frac{\varepsilon}{4} \\
&= \frac{\varepsilon}{4} + \left|\int_{c+ia_N}^{d+ia_N} f(s)ds + \int_{d+ia_N}^{d-ia_N} f(s)ds + \int_{d-ia_N}^{c-ia_N} f(s)ds\right| + \frac{\varepsilon}{4}
\end{aligned}
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&\leq \frac{\varepsilon}{4} + \left|\int_{c+ia_N}^{d+ia_N} f(s)ds\right| + \left|\int_{d+ia_N}^{d-ia_N} f(s)ds\right| + \left|\int_{d-ia_N}^{c-ia_N} f(s)ds\right| + \frac{\varepsilon}{4}
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Let $\varepsilon > 0$. Choose $N \geq N_1, N_2, N_3$

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2\pi|L(u) - u^{-\Delta}\Pi(u)| &= \left| \int_{c-i\infty}^{c+i\infty} f(s)ds - 2\pi i \sum_{n\in\mathbb{Z}} \mathrm{r}(s_n) \right| \\
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&= \frac{\varepsilon}{4} + \left| \int_{c-ia_N}^{c+ia_N} f(s)ds - \int_{\Gamma_N} f(s)ds \right| + \frac{\varepsilon}{4} \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \left| \int_{d+ia_N}^{d-ia_N} f(s)ds \right| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}
\end{aligned}
$$

Let $\varepsilon > 0$. Choose $N \geq N_1, N_2, N_3, N_4$.

$$
\begin{aligned}
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&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{K_d}{u^d} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}
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&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \left| \int_{d+ia_N}^{d-ia_N} f(s)ds \right| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{K_d}{u^d} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{K_d}{u^d} + \varepsilon
\end{aligned}
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Step 3: Residue Theorem (Recall: bounding $\left|L(n) - n^{-\Delta}\Pi(n)\right|$)

Let $\varepsilon > 0$. Choose $N \geq N_1, N_2, N_3, N_4$.

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&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{K_d}{u^d} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{K_d}{u^d} + \varepsilon
\end{aligned}
$$

Choosing $d = \Delta + 1$ gives $|L(n) - n^{-\Delta}\Pi(n)| \leq \frac{K_{\Delta+1}/2\pi}{n^{\Delta+1}} = n^{-\Delta}\frac{C_2}{n}$.

Summary

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We have shown that

$$\left| M_n - n^{-\Delta}\Pi(n) \right| \leq |M_n - L(n)| + \left| L(n) - n^{-\Delta}\Pi(n) \right|$$

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We have shown that

$$\begin{aligned} \left| M_n - n^{-\Delta}\Pi(n) \right| &\le |M_n - L(n)| + \left| L(n) - n^{-\Delta}\Pi(n) \right| \\ &\le n^{-\Delta}\frac{C_1}{n} + n^{-\Delta}\frac{C_2}{n} \\ &\le n^{-\Delta}\frac{C}{n} \qquad \blacksquare \end{aligned}$$

Summary

We have shown that

$$\left|M_n - n^{-\Delta}\Pi(n)\right| \leq |M_n - L(n)| + \left|L(n) - n^{-\Delta}\Pi(n)\right| \leq n^{-\Delta}\frac{C_1}{n} + n^{-\Delta}\frac{C_2}{n} \leq n^{-\Delta}\frac{C}{n} \qquad \blacksquare$$

Questions?