$\frac{3}{2}$-Generation of Finite Groups

Scott Harper

(University of Bristol)

Cambridge Junior Algebra Seminar

4th March 2016

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Are other important classes of groups 2-generated?

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Sporadic groups	Aschbacher & Guralnick, 1984

∎

Netto's Conjecture

If we arbitrarily select two or more substitutions of n elements, it is to be regarded as extremely probable that the group of lowest order which contains these is the symmetric group, or at least the alternating group. In the case of two substitutions the probability in favor of the symmetric group may be taken as about $\frac{3}{4}$, and in favor of the alternating, but not symmetric, group as about $\frac{1}{4}$. In order that any given substitutions may generate a group which is only a part of the $n!$ possible substitutions, very special relations are necessary, and it is highly improbable that arbitrarily chosen substitutions $s_i = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ x_{i_1} & x_{i_2} & \ldots & x_{i_n} \end{pmatrix}$ should satisfy these conditions. The exception most likely to occur would be that all the given substitutions were severally equivalent to an even number of transpositions and would consequently generate the alternating group.

E. Netto, *The theory of substitutions and its application to algebra*,
Trans. F. N. Cole, Ann Arbor, Michigan, (1892)

Simple Groups: It is possible; is it probable?

Let G be a finite simple group.

Write

$$P(G) = \frac{|\{(x,y) \in G \times G \mid \langle x,y \rangle = G\}|}{|G|^2}.$$

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Lower Bound

Menezes, Quick & Roney-Dougal, 2013

$P(G) \geq \frac{53}{90}$ with equality if and only if $G = A_6$.

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A group G has spread k if for any distinct $x_1, \ldots, x_k \in G \setminus 1$ there exists an element $z \in G$ such that $\langle x_1, z \rangle = \cdots = \langle x_k, z \rangle = G$.

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Write $s(G)$ for the greatest integer k such that G has spread k.

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Theorem (Breuer, Guralnick & Kantor, 2008)

For a finite simple group G, $s(G) \geq 2$.

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Conjecture (Breuer, Guralnick & Kantor, 2008)

A finite group is $\frac{3}{2}$-generated iff every proper quotient is cyclic.

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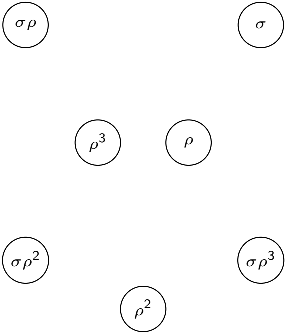

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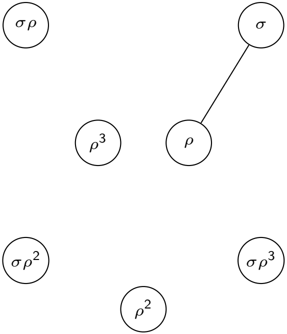

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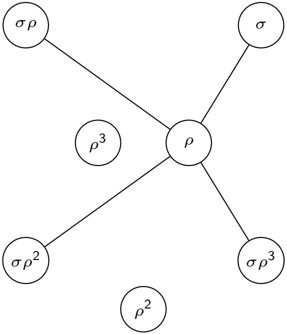

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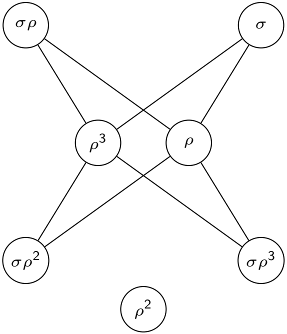

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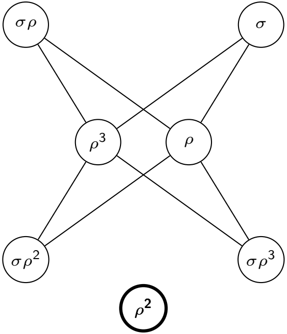

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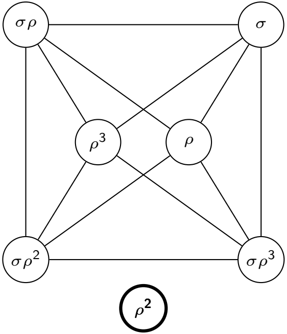

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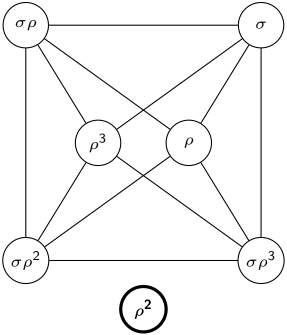

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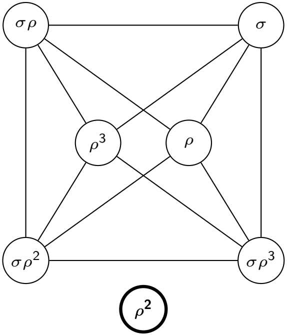

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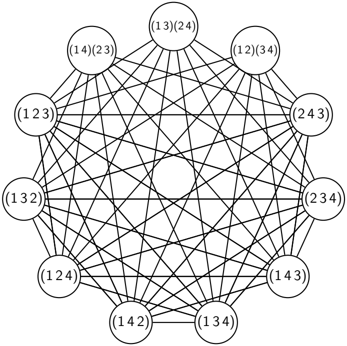

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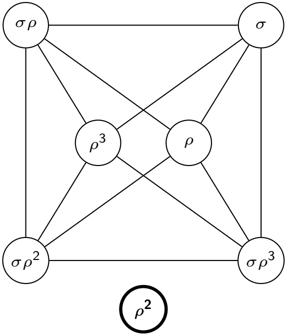

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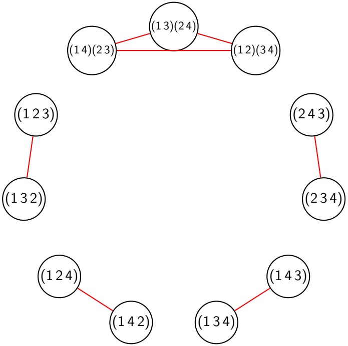

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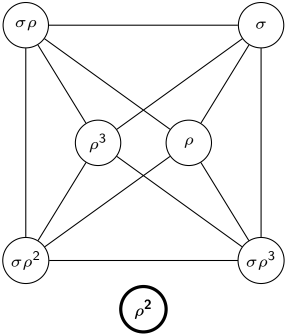

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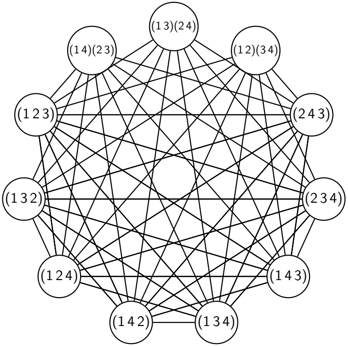

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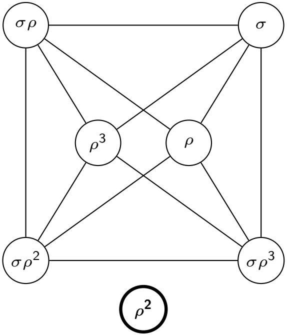

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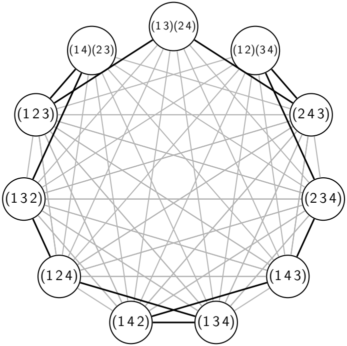

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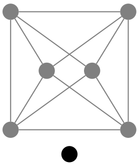
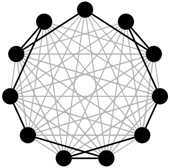


- $\Gamma(G)$ is disconnected and, moreover, has isolated vertices (i.e. $s(G) = 0$).



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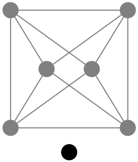
Generating Graphs



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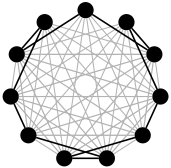

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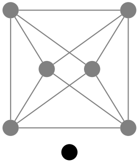

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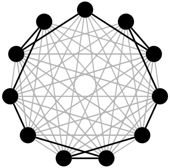

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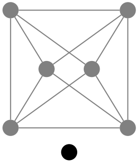

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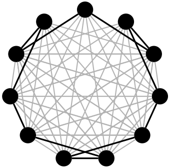

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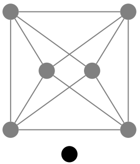

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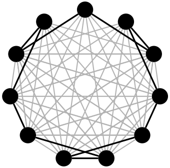

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- Chromatic number, clique number, coclique number ...?
- When does $\Gamma(G)$ determine G?

Which groups are $\frac{3}{2}$-generated?

Main Conjecture

A finite group G is $\frac{3}{2}$-generated iff every proper quotient of G is cyclic.

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Examples: $G = S_n$ (with $T = A_n$); $G = PGL_n(q)$ (with $T = PSL_n(q)$).

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Problem

Classical groups	Symplectic groups, Orthogonal groups, Unitary groups
Exceptional groups	

Uniform Spread

A group G has spread k if for any distinct $x_1, \dots, x_k \in G \setminus 1$ there exists an element $z \in G$ such that $\langle x_1, z \rangle = \cdots = \langle x_k, z \rangle = G$.

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Many of the earlier results on spread, in fact, established uniform spread.

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Let $s \in G$. Write

$$P(x,s) = \frac{|\{z \in s^G \mid \langle x, z\rangle \neq G\}|}{|s^G|}.$$

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Lemma 2

$$P(x,s) \leq \sum_{H \in \mathcal{M}(G,s)} \frac{|x^G \cap H|}{|x^G|}.$$

Example: Alternating Group A_5

Proposition

The alternating group A_5 has uniform spread (at least) two.

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Then $\mathcal{M}(A_5, s) = \{H\}$ where $H \cong D_{10}$.

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x has order 5 $\;|x^G| = 12$

Example: Alternating Group A_5

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Example: Alternating Group A_5

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Conjugacy classes of $G = A_5$:

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Which groups are $\frac{3}{2}$-generated?

Main Conjecture

A finite group is $\frac{3}{2}$-generated iff every proper quotient is cyclic.

Strategy: Show that $\langle T, g \rangle$ is $\frac{3}{2}$-generated for T simple and $g \in \mathrm{Aut}(T)$.

Progress so far

Alternating groups	Symmetric groups: Brenner & Wiegold, 1975 & 1980
	Extensions of A_6: Computation
Sporadic groups	Breuer, Guralnick & Kantor, 2008
Classical groups	Linear groups: Burness & Guest, 2013

Problem

Classical groups	Symplectic groups, Orthogonal groups, Unitary groups
Exceptional groups	

Symplectic Groups

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Theorem (H, 2016)

For q even and $n \geq 6$, $Sp_n(q){:}\langle \sigma^i \rangle$ is $\frac{3}{2}$-generated.

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Let G be a classical almost simple group with socle T. Any maximal subgroup of G which does not contain T belongs to one of:

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- The order of s^e is divisible by $q_0^{\frac{n-2}{2}} + 1$.

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Use very general results. For example, by a theorem of Burness (2007),

$$|x^G \cap H| < |x^G|^{\varepsilon}$$

for $\varepsilon \approx \frac{1}{2}$, provided that H is not in $\mathcal{C}_1$.

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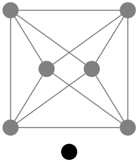

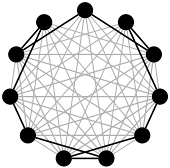

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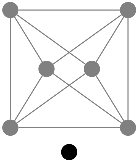

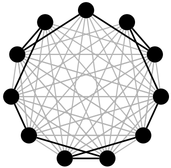

Conjecture

The following are equivalent.

- Every proper quotient of G is cyclic.
- $\Gamma(G)$ has no isolated vertices (i.e. G has spread one).
- $\Gamma(G)$ has diameter two (i.e. G has spread two).
- $\Gamma(G)$ has a Hamiltonian cycle.