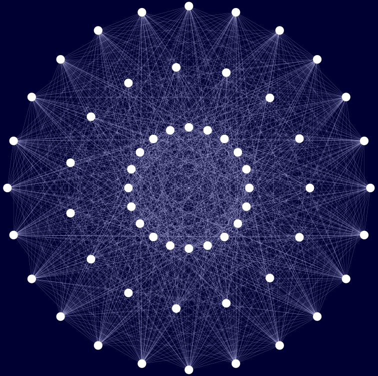
$\frac{3}{2}$-Generation of Finite Groups



Scott Harper

(University of Bristol)

Postgraduate Group Theory Conference

30th June 2016

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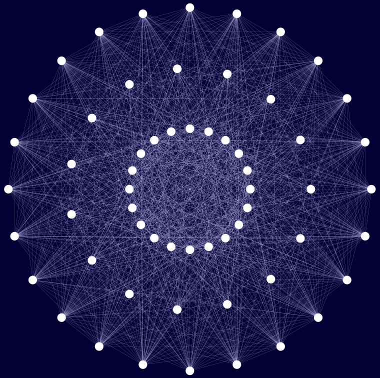

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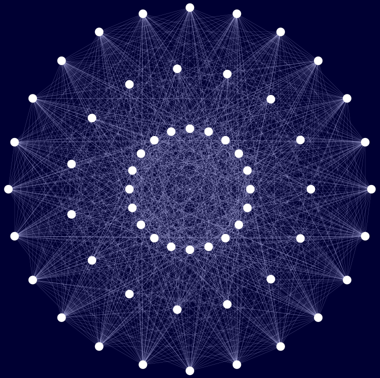

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Theorem (Steinberg 1962; Aschbacher & Guralnick 1984)

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Any more? Groups such that all proper quotients are cyclic?

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Conjecture (Breuer, Guralnick & Kantor, 2008)

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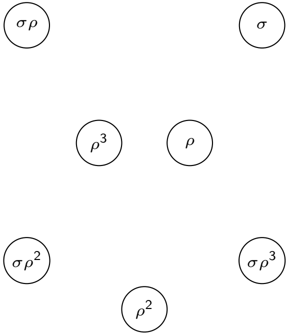

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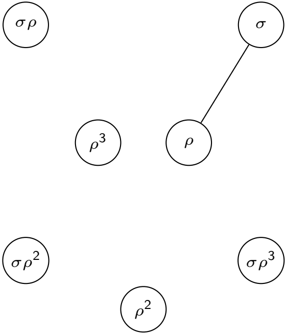

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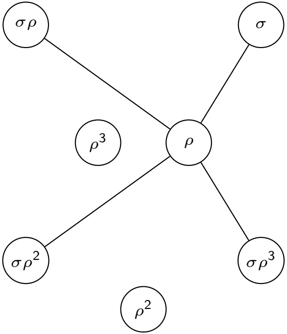

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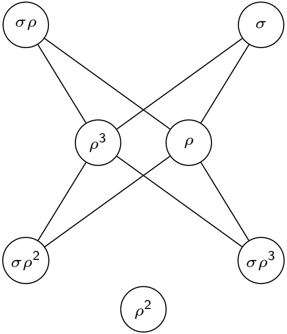

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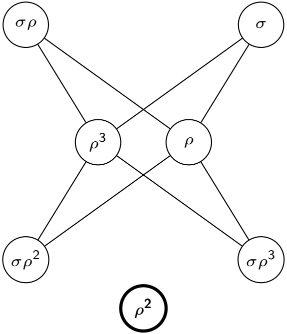

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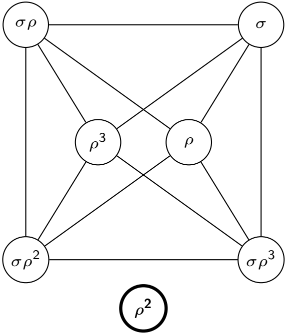

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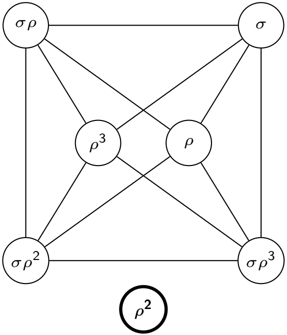

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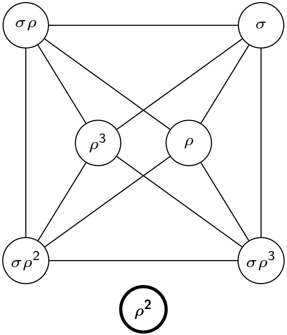

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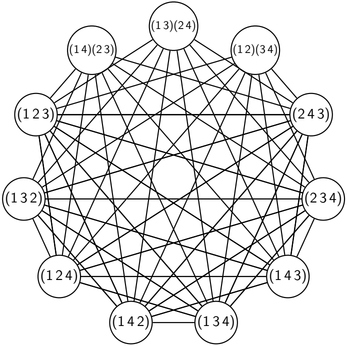

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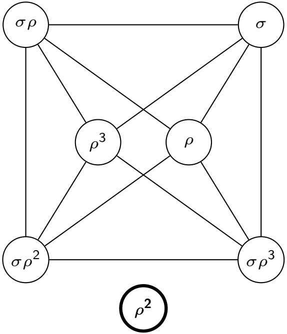

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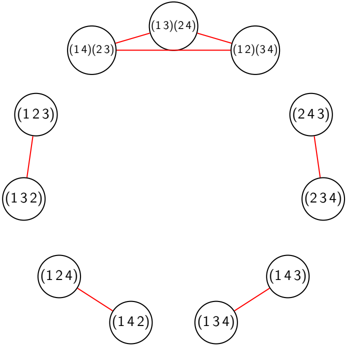

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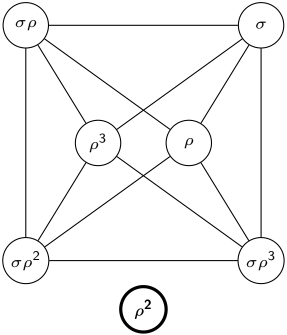

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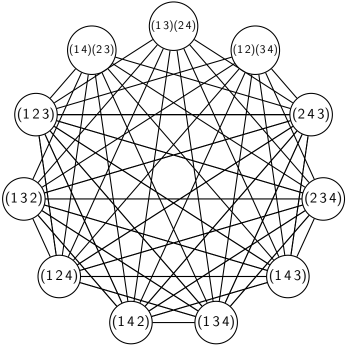

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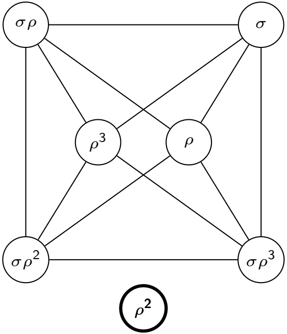

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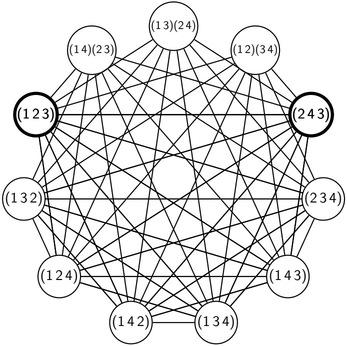

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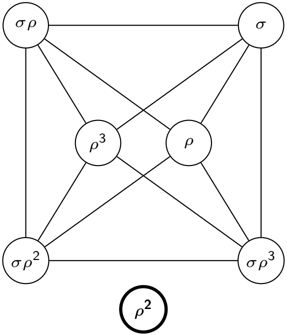

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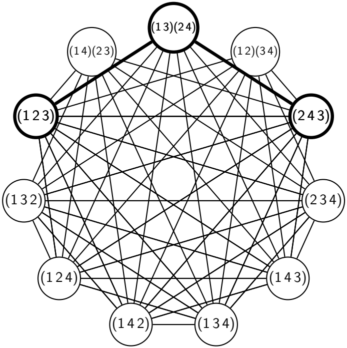

Spread and Uniform Spread

A group G has spread k if for any distinct $x_1, \ldots, x_k \in G \setminus 1$ there exists an element $z \in G$ such that $\langle x_1, z \rangle = \cdots = \langle x_k, z \rangle = G$.

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A group G has uniform spread k if there exists a conjugacy class C such that for any distinct $x_1, \dots, x_k \in G \setminus 1$ there exists an element $z \in C$ such that $\langle x_1, z \rangle = \cdots = \langle x_k, z \rangle = G$.

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Examples: $G = S_n$ (with $T = A_n$); $G = PGL_n(q)$ (with $T = PSL_n(q)$).

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Project: Show $\langle T, g \rangle$ has strong spread properties when T is of Lie type.

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Let $s \in G$. Write

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So $G = Sp_n(q){:}\langle \sigma^i \rangle$.

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For each $z \in Sp_n(q_0)$, $z = a^{-1}s^e a$ for some $s \in Sp_n(q)\sigma^i$ and $a \in Sp_n(\overline{\mathbb{F}_q})$.

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Choose s such that

$$s^e = \left(\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right) \in Sp_n(q_0)$$

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Theorem (Aschbacher, 1984)

Let G be a classical almost simple group with socle T. Any maximal subgroup of G which does not contain T belongs to one of:

- $\mathcal{C}_1, \ldots, \mathcal{C}_8$ *(a family of geometric subgroups);*
- $\mathcal{S}$ *(the family of almost simple irreducible subgroups).*

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3. Calculate the probability $P(x,s)$.

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Use very general results. For example, by a theorem of Burness (2007),

$$|x^G \cap H| < |x^G|^{\varepsilon}$$

for $\varepsilon \approx \frac{1}{2}$, provided that H is not in $\mathcal{C}_1$.

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Theorem (H, 2016)

Let $n \neq 4$ and write $G = \langle T, g \rangle$ where $T = PSp_n(q)$ and $g \in \mathrm{Aut}(T)$. Then $u(G) \geq 2$ and $u(G) \to \infty$ as $q \to \infty$.

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Question: Is there a finite group with spread exactly one?