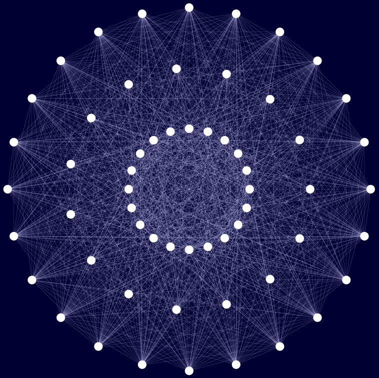
Generating Graphs of Finite Groups



Scott Harper

(University of Bristol)

LMS Graduate Meeting

8th July 2016

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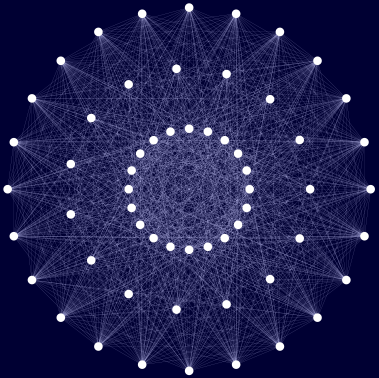

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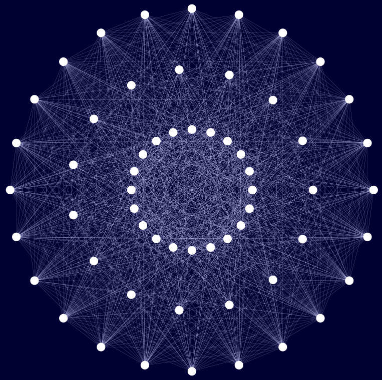

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Are other important families of finite groups 2-generated?

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Theorem (Steinberg, 1962; Aschbacher & Guralnick, 1984)

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Netto's Conjecture

If we arbitrarily select two or more substitutions of n elements, it is to be regarded as extremely probable that the group of lowest order which contains these is the symmetric group, or at least the alternating group. In the case of two substitutions the probability in favor of the symmetric group may be taken as about $\frac{3}{4}$, and in favor of the alternating, but not symmetric, group as about $\frac{1}{4}$. In order that any given substitutions may generate a group which is only a part of the $n!$ possible substitutions, very special relations are necessary, and it is highly improbable that arbitrarily chosen substitutions $s_i = \begin{pmatrix} x_1 \, x_2 \, \ldots \, x_n \\ x_{i_1} \, x_{i_2} \, \ldots \, x_{i_n} \end{pmatrix}$ should satisfy these conditions. The exception most likely to occur would be that all the given substitutions were severally equivalent to an even number of transpositions and would consequently generate the alternating group.

E. Netto, *The theory of substitutions and its application to algebra*, Trans. F. N. Cole, Ann Arbor, Michigan, (1892)

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Theorem (Menezes, Quick & Roney-Dougal, 2013)

If G is simple then $P(G) \geq \frac{53}{90}$ with equality if and only if $G = A_6$.

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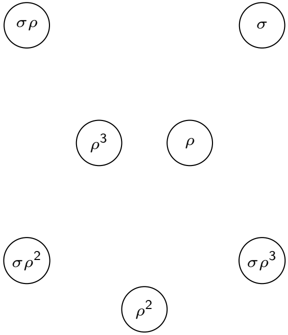

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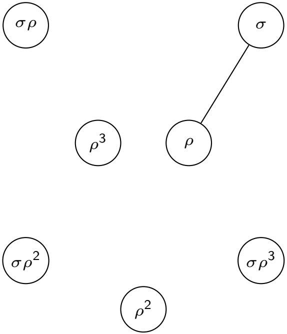

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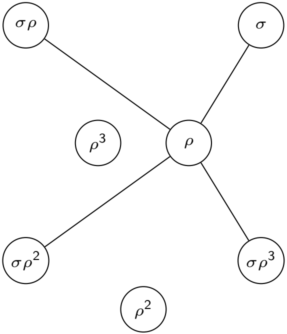

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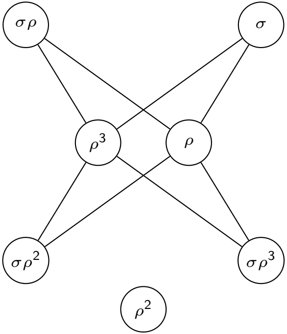

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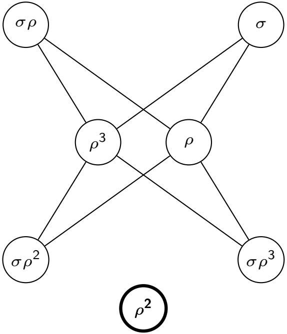

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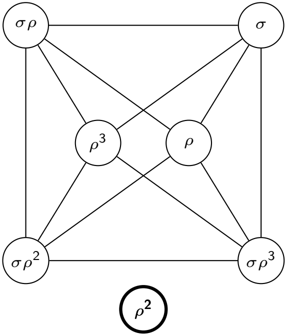

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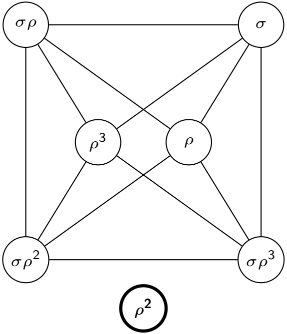

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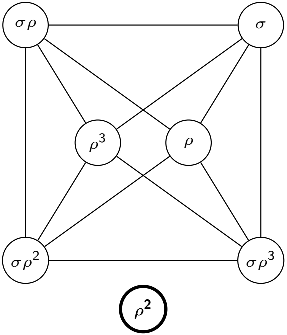

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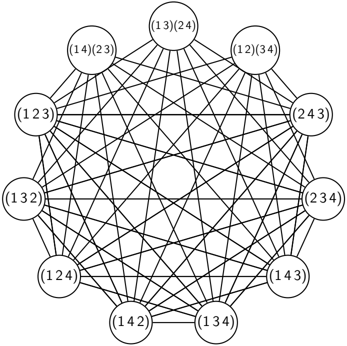

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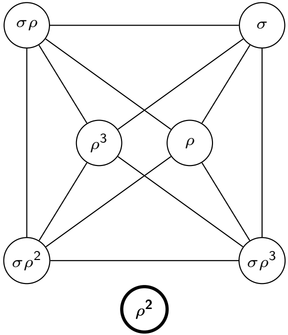

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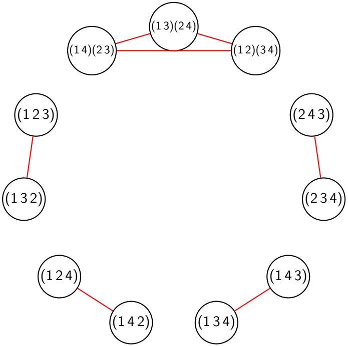

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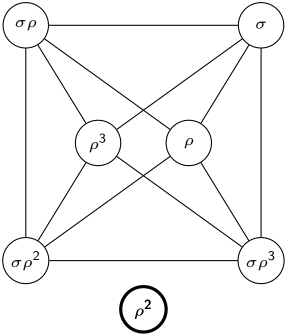

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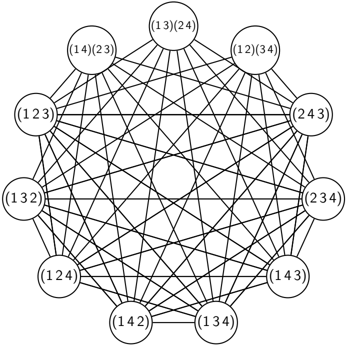

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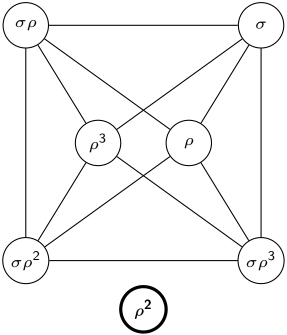

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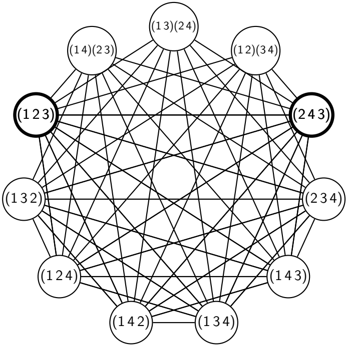

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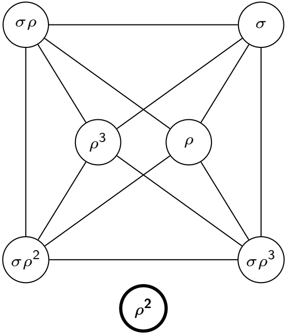

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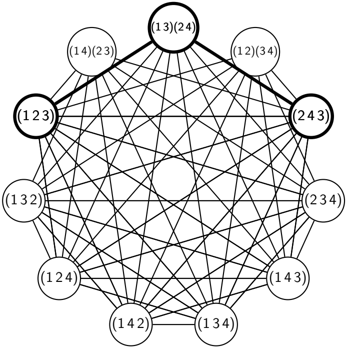

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Any more? Groups such that all proper quotients are cyclic?

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If $\Gamma(G)$ *has no isolated vertices then every proper quotient of* G *is cyclic.*

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Conjecture

The following are equivalent

- Every proper quotient of G is cyclic.
- $\Gamma(G)$ has no isolated vertices.
- $\Gamma(G)$ has diameter two.

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For a prime power q and even n, the symplectic group $Sp_n(q) \leq GL_n(q)$ is the stabiliser of a non-degenerate alternating form on $\mathbb{F}_q^n$.

Theorem (H, 2016)

Let $n \neq 4$ and write $G = \langle T, g\rangle$ where $T = PSp_n(q)$ and $g \in \mathrm{Aut}(T)$. Then $\Gamma(G)$ has diameter two.

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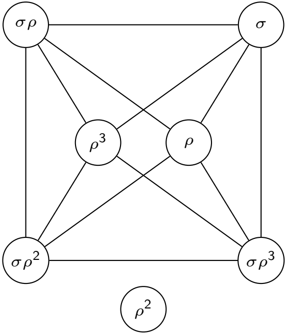

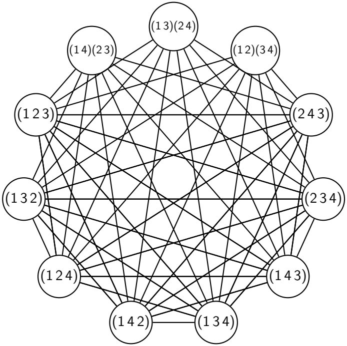

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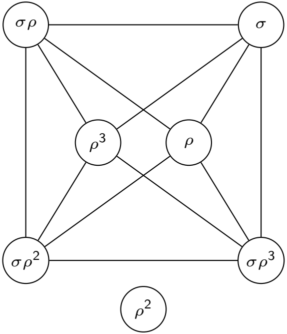

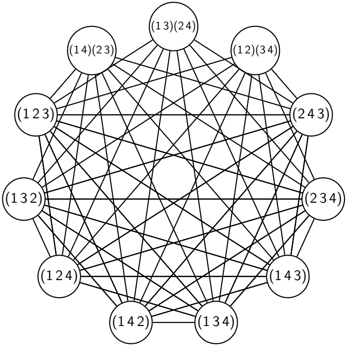

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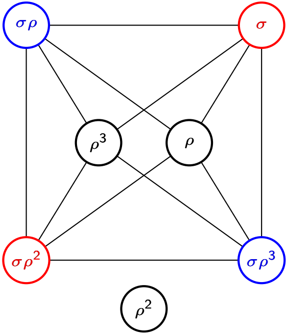

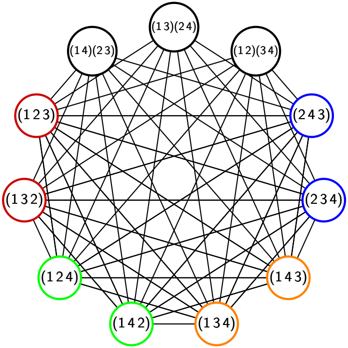

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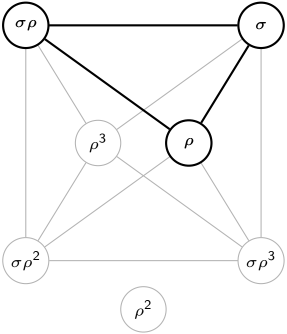

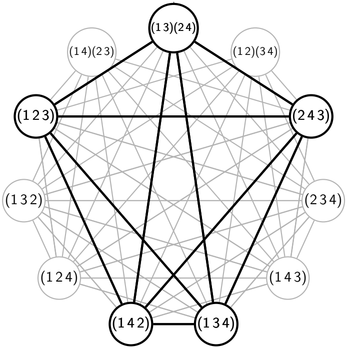

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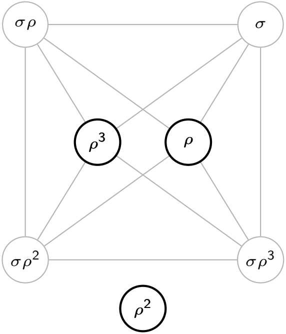

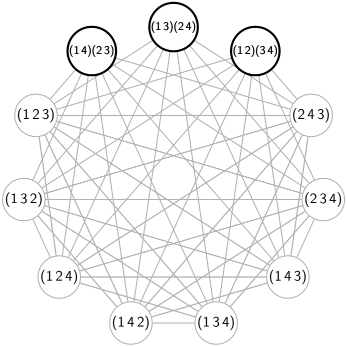

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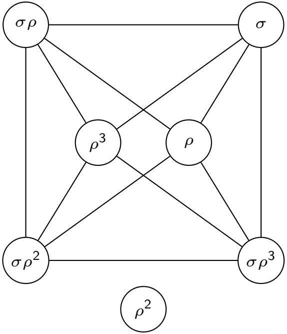

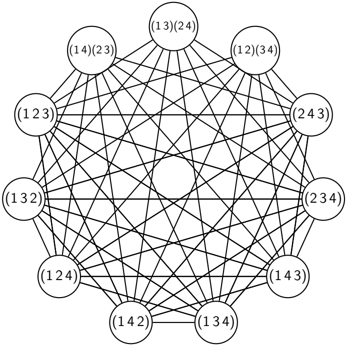

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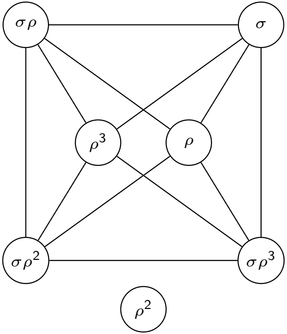

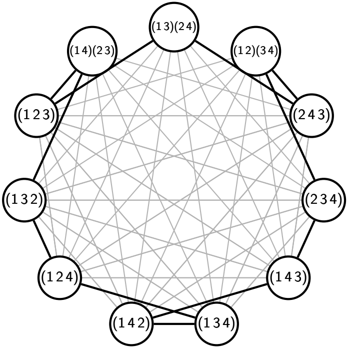

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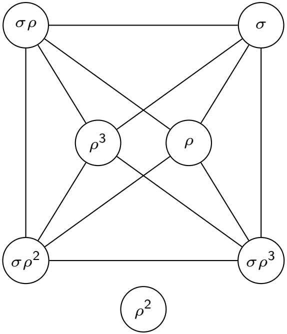

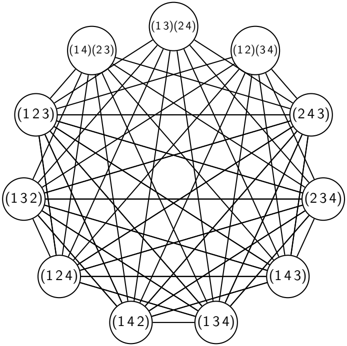

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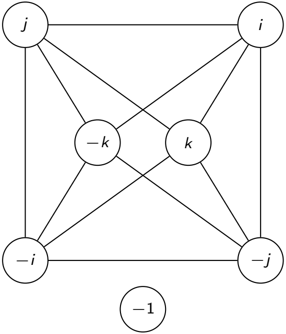
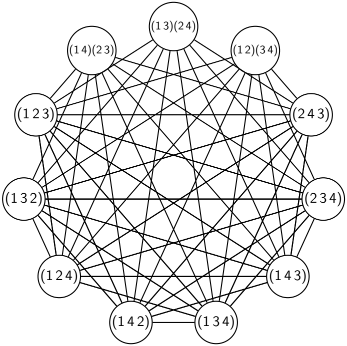
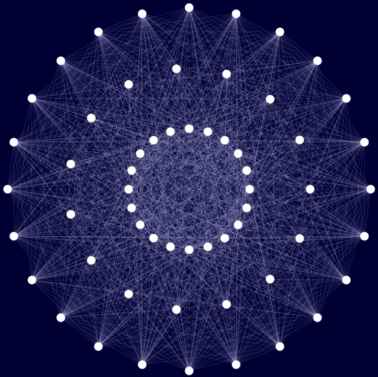




Generating Graphs of Finite Groups



Scott Harper

(University of Bristol)

LMS Graduate Meeting

8th July 2016