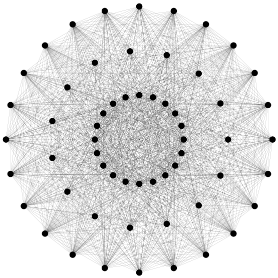
$\frac{3}{2}$-Generation of Finite Groups

Scott Harper



Pure Postgraduate Seminar

10th March 2017

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Alternating groups are 2-generated:

- if n is odd $A_n = \langle (1\,2\,3), (1\,2\,\ldots\,n) \rangle$
- if n is even $A_n = \langle (1\,2\,3), (2\,3\,\ldots\,n) \rangle$

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and do this so that $q_1, \ldots, q_k$ and n_k are all prime.

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Theorem For $n \geq 5$, the alternating group A_n is simple.

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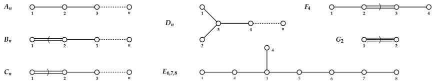
The Periodic Table Of Finite Simple Groups

0, C_1, $\mathbb{Z}_1$
1
1

Dynkin Diagrams of Simple Lie Algebras



$A_1(4)$, $A_1(5)$ A_5 60	$A_2(2)$ $A_1(7)$ 168											$^2A_3(4)$ $B_2(3)$ 25 920	$C_3(3)$ 4 585 351 680	$D_4(2)$ 174 182 400	$^2D_4(2^2)$ 197 406 720	$G_2(2)'$ $^2A_2(9)$ 6 048
$A_1(9)$, $B_2(2)'$ A_6 360	$^2G_2(3)'$ $A_1(8)$ 504											$B_2(4)$ 979 200	$C_3(5)$ 228 501 000 000 000	$D_4(3)$ 4 952 179 814 400	$^2D_4(3^2)$ 10 151 968 619 520	$^2A_2(16)$ 62 400
A_7 2 520	$A_1(11)$ 660	$E_6(2)$ 214 841 575 522 005 575 270 400	$E_7(2)$ 7 997 476 042 075 799 759 100 487 262 680 802 918 400	$E_8(2)$ 337 804 753 143 634 806 261 388 190 614 085 595 079 991 692 242 467 651 576 160 959 909 068 800 000	$F_4(2)$ 3 311 126 603 366 400	$G_2(3)$ 4 245 696	$^3D_4(2^3)$ 211 341 312	$^2E_6(2^2)$ 76 532 479 683 774 853 939 200	$^2B_2(2^3)$ 29 120	Tits* $^2F_4(2)'$ 17 971 200	$^2G_2(3^3)$ 10 073 444 472	$B_3(2)$ 1 451 520	$C_4(3)$ 65 784 756 654 489 600	$D_5(2)$ 23 499 295 948 800	$^2D_5(2^2)$ 25 015 379 558 400	$^2A_2(25)$ 126 000
$A_3(2)$ A_8 20 160	$A_1(13)$ 1 092	$E_6(3)$ [illegible]	$E_7(3)$ 1 271 375 236 818 136 742 240 479 751 139 021 644 554 379 203 770 766 254 617 395 200	$E_8(3)$ [illegible]	$F_4(3)$ 5 734 420 792 816 671 844 761 600	$G_2(4)$ 251 596 800	$^3D_4(3^3)$ 20 560 831 566 912	$^2E_6(3^2)$ 14 636 855 916 969 695 633 965 120 680 532 377 600	$^2B_2(2^5)$ 32 537 600	$^2F_4(2^3)$ 264 905 352 699 586 176 614 400	$^2G_2(3^5)$ 49 825 657 439 340 552	$B_2(5)$ 4 680 000	$C_3(7)$ 273 457 218 604 953 600	$D_4(5)$ 8 911 539 000 000 000 000	$^2D_4(4^2)$ 67 536 471 195 648 000	$^2A_3(9)$ 3 265 920
A_9 181 440	$A_1(17)$ 2 448	$E_6(4)$ 85 528 710 781 342 640 103 833 619 055 142 765 466 746 880 000	$E_7(4)$ [illegible]	$E_8(4)$ [illegible]	$F_4(4)$ 19 009 825 523 840 945 451 297 669 120 000	$G_2(5)$ 5 859 000 000	$^3D_4(4^3)$ 67 802 350 642 790 400	$^2E_6(4^2)$ 85 696 576 147 617 709 485 896 772 387 584 983 695 360 000 000	$^2B_2(2^7)$ 34 093 383 680	$^2F_4(2^5)$ 1 318 633 155 799 591 447 702 161 609 782 722 560 000	$^2G_2(3^7)$ 239 189 910 264 352 349 332 632	$B_2(7)$ 138 297 600	$C_3(9)$ 54 025 731 402 499 584 000	$D_5(3)$ 1 289 512 799 941 305 139 200	$^2D_4(5^2)$ 17 880 203 250 000 000 000	$^2A_2(64)$ 5 515 776
A_n $\frac{n!}{2}$	$PSL_{n+1}(q)$, $L_{n+1}(q)$ $A_n(q)$ $\frac{q^{n(n+1)/2}}{(n+1,q-1)}\prod_{i=1}^{n}(q^{i+1}-1)$	$E_6(q)$ $\frac{q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)}{(3,q-1)}$	$E_7(q)$ $\frac{q^{63}}{(2,q-1)}\prod_{i=1, i\neq 2,8}^{9}(q^{2i}-1)$	$E_8(q)$ $q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	$F_4(q)$ $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	$G_2(q)$ $q^6(q^6-1)(q^2-1)$	$^3D_4(q^3)$ $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	$^2E_6(q^2)$ $\frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{(3,q+1)}$	$^2B_2(2^{2n+1})$ $q^2(q^2+1)(q-1)$	$^2F_4(2^{2n+1})$ $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	$^2G_2(3^{2n+1})$ $q^3(q^3+1)(q-1)$	$O_{2n+1}(q)$, $\Omega_{2n+1}(q)$ $B_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$PSp_{2n}(q)$ $C_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$O_{2n}^+(q)$ $D_n(q)$ $\frac{q^{n(n-1)}(q^n-1)}{(4,q^n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$O_{2n}^-(q)$ $^2D_n(q^2)$ $\frac{q^{n(n-1)}(q^n+1)}{(4,q^n+1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$PSU_{n+1}(q)$ $^2A_n(q^2)$ $\frac{q^{n(n+1)/2}}{(n+1,q+1)}\prod_{i=2}^{n+1}(q^i-(-1)^i)$

C_2 2
C_3 3
C_5 5
C_7 7
C_{11} 11
C_{13} 13
$\mathbb{Z}_p$ C_p p

- Alternating Groups
- Classical Chevalley Groups
- Chevalley Groups
- Classical Steinberg Groups
- Steinberg Groups
- Suzuki Groups
- Ree Groups and Tits Group*
- Sporadic Groups
- Cyclic Groups

Alternates†
Symbol
Order‡

M_{11} 7 920	M_{12} 95 040	M_{22} 443 520	M_{23} 10 200 960	M_{24} 244 823 040	$J(1)$, $J(11)$ J_1 175 560	HJ J_2 604 800	HJM J_3 50 232 960	J_4 86 775 571 046 077 562 880	HS 44 352 000	McL 898 128 000	F_7, HHM, HTH He 4 030 387 200	Ru 145 926 144 000
Sz Suz 448 345 497 600	$O'NS$, O–S $O'N$ 460 815 505 920	$\cdot 3$ Co_3 495 766 656 000	$\cdot 2$ Co_2 42 305 421 312 000	$\cdot 1$ Co_1 4 157 776 806 543 360 000	F_5, D HN 273 030 912 000 000	LyS Ly 51 765 179 004 000 000	F_3, E Th 90 745 943 887 872 000	$M(22)$ Fi_{22} 64 561 751 654 400	$M(23)$ Fi_{23} 4 089 470 473 293 004 800	F_{3+}, $M(24)'$ Fi'_{24} 1 255 205 709 190 661 721 292 800	F_2 B 4 154 781 481 226 426 191 177 580 544 000 000	F_1, M_1 M 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000

*The Tits group $^2F_4(2)'$ is not a group of Lie type, but is the (index 2) commutator subgroup of $^2F_4(2)$. It is usually given honorary Lie type status.

The groups starting on the second row are the classical groups. The sporadic suzuki group is unrelated to the families of Suzuki groups.

†For sporadic groups and families, alternate names in the upper left are other names by which they may be known. For specific non-sporadic groups these are used to indicate isomorphims. All such isomorphisms appear on the table except the family $B_n(2^m) \cong C_n(2^m)$.

‡Finite simple groups are determined by their order with the following exceptions:
$B_n(q)$ and $C_n(q)$ for q odd, $n > 2$;
$A_8 \cong A_3(2)$ and $A_2(4)$ of order 20160.

https://irandrus.files.wordpress.com/2012/06/periodic-table-of-groups.pdf

Theorem (The Classification of Finite Simple Groups)

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Caveat mathematice

There are many ways of putting simple groups together.

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Theorem (Steinberg, 1962)

Every finite simple group is 2-generated.

What are the chances?

If we arbitrarily select two or more substitutions of n elements, it is to be regarded as extremely probable that the group of lowest order which contains these is the symmetric group, or at least the alternating group. In the case of two substitutions the probability in favor of the symmetric group may be taken as about $\frac{3}{4}$, and in favor of the alternating, but not symmetric, group as about $\frac{1}{4}$. In order that any given substitutions may generate a group which is only a part of the $n!$ possible substitutions, very special relations are necessary, and it is highly improbable that arbitrarily chosen substitutions $s_i = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ x_{i_1} & x_{i_2} & \ldots & x_{i_n} \end{pmatrix}$ should satisfy these conditions. The exception most likely to occur would be that all the given substitutions were severally equivalent to an even number of transpositions and would consequently generate the alternating group.

E. Netto, *The theory of substitutions and its application to algebra*, Trans. F. N. Cole, Ann Arbor, Michigan, (1892)

Let $P(G)$ be the probability that two random elements generate G.

That is,

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Numerical evidence

n	5	6	7	8	9	10
$P(A_n)$	0.633	0.588	0.726	0.738	0.848	0.875

GAP computations by N. Menezes, M. Quick and C. M. Roney-Dougal

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Using the work of Erdős and Turán, Dixon proved Netto's conjecture.

Theorem (Dixon, 1969)

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If G is simple then $P(G) \geq \frac{53}{90}$ with equality if and only if $G = A_6$.

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Summary: It's easy to generate a finite simple group with two elements.

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Any more? Groups such that all proper quotients are cyclic?

Proposition

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Conjecture (Breuer, Guralnick & Kantor, 2008)

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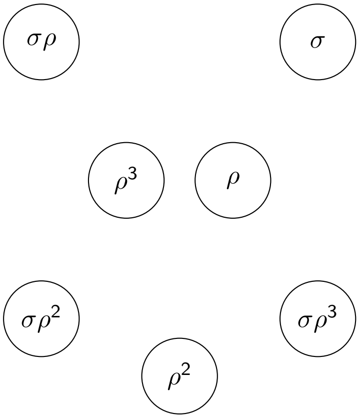

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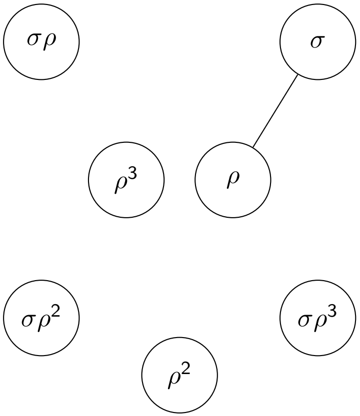

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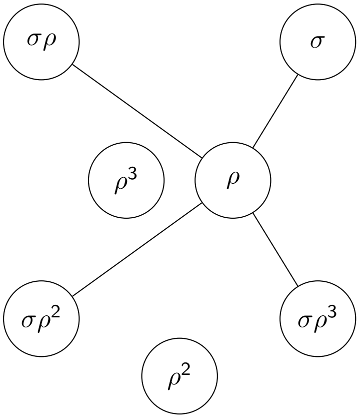

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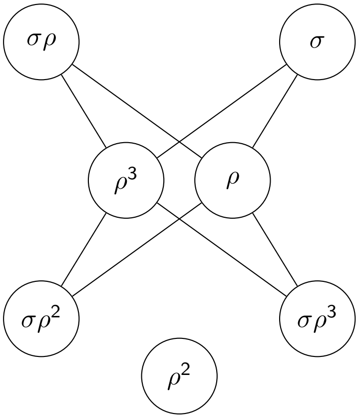

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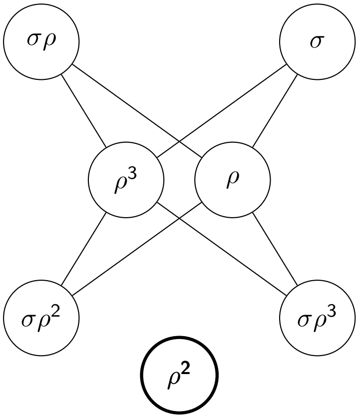

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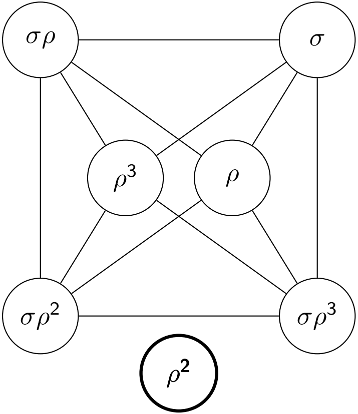

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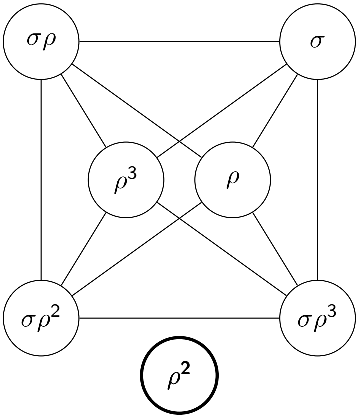

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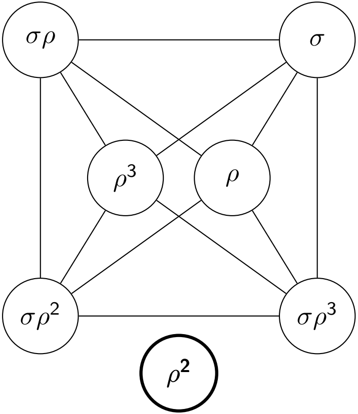

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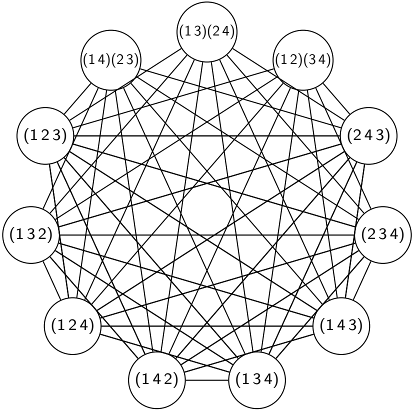

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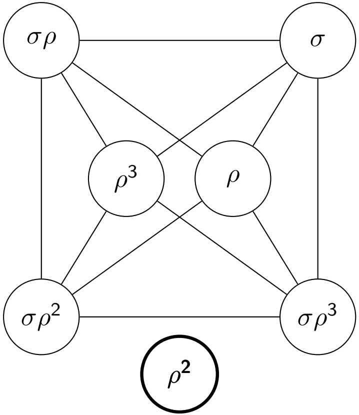

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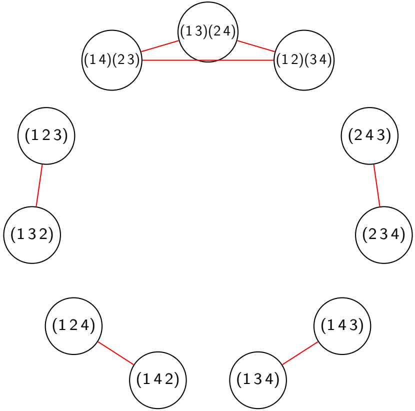

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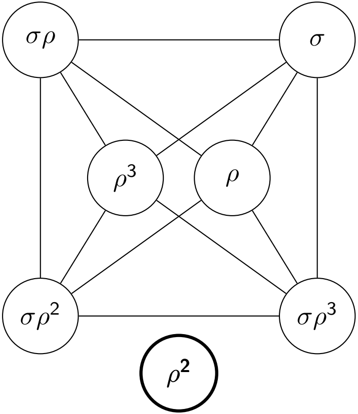

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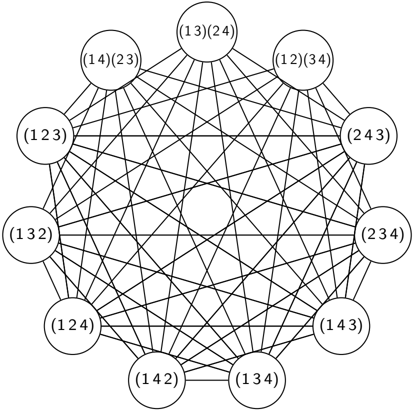

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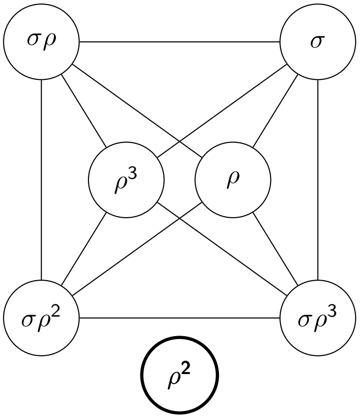

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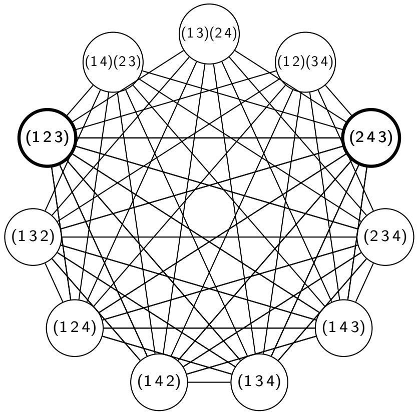

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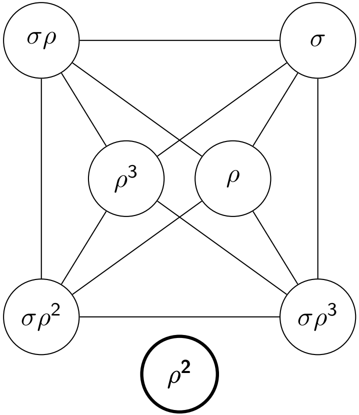

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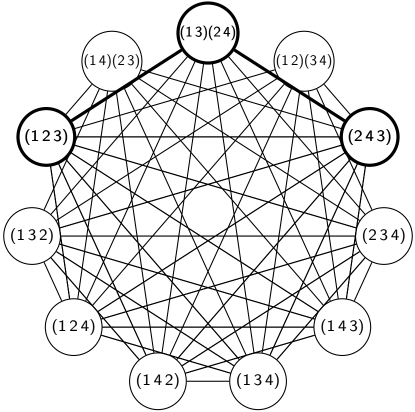

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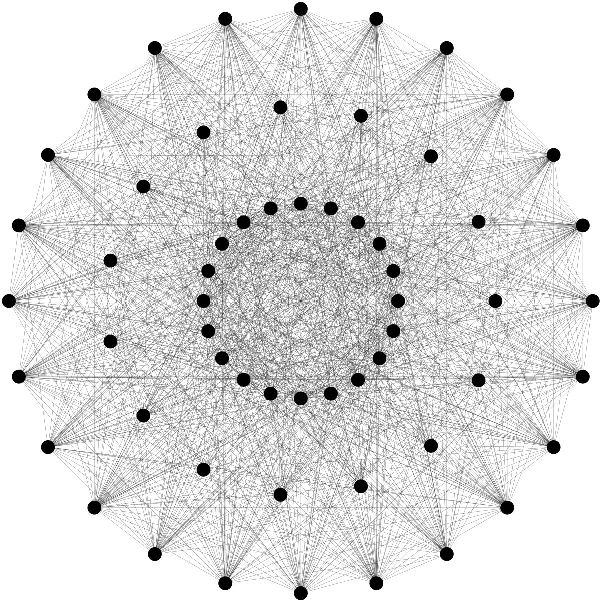

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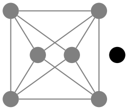

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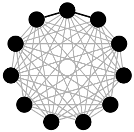

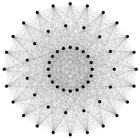

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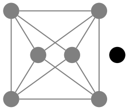

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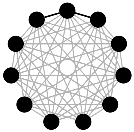

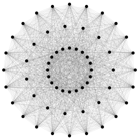

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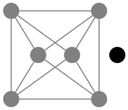

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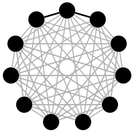

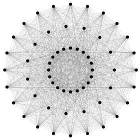

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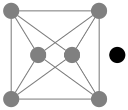

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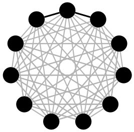

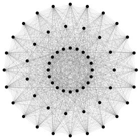

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Theorem (Breuer, Guralnick & Kantor, 2008)

Every finite simple group G has (at least) spread two.

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Main Conjecture

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Examples: $G = S_n$ (with $T = A_n$); $G = \mathrm{PGL}_n(q)$ (with $T = \mathrm{PSL}_n(q)$).

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Project: Show $\langle T, g \rangle$ has strong spread properties when T is of Lie type.

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The alternating group A_5 has spread two.

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Proof

1. Choose $s \in A_5$ and determine $\mathcal{M}(G, s)$

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Choose $s = (1\,2\,3\,4\,5)$.

Then $\mathcal{M}(A_5, s) = \{H\}$ where $H \cong D_{\pentagon}$

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In all cases $P(x, s) < \frac{1}{2}$. So $s(A_5) \geq 2$. ■

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Exceptional groups of Lie type

Chevalley	$E_6(q), E_7(q), E_8(q), F_4(q), G_2(q)$
Steinberg	${}^3D_4(q)$, ${}^2E_6(q)$
Suzuki and Ree	${}^2B_2(q)$, ${}^2F_4(q)$, ${}^2G_2(q)$

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Odd-Dimensional Orthogonal Groups

Let $n = 2m + 1$ and let f be a non-deg. symmetric bilinear form on V.

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Let $G_n = \langle T_n, g_n \rangle$ for $T_n \in \mathcal{T}$ and $g_n \in \mathrm{Aut}(T_n)$. Assume that $|G_n| \to \infty$.

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Results on Spread

Write $\mathcal{T} = \{\mathrm{PSp}_{2m}(q) \mid m \geq 2, q \neq 2\} \cup \{\Omega_{2m+1}(q) \mid m \geq 3\}$.

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- odd-dimensional orthogonal groups, or
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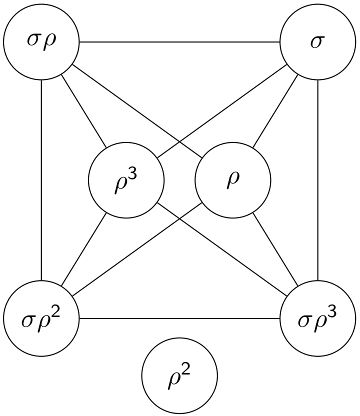

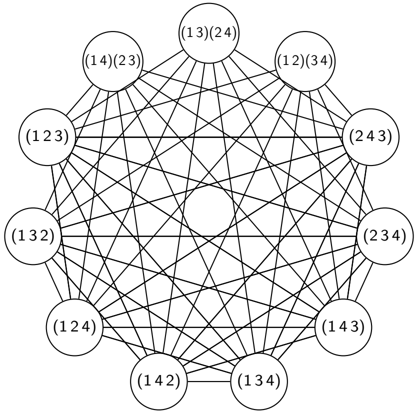

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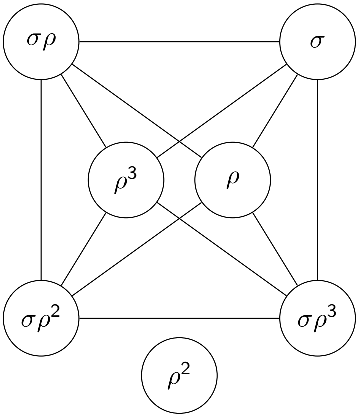

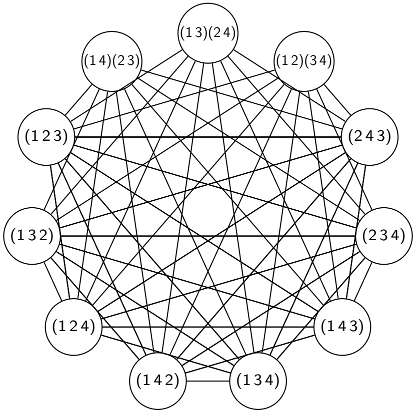

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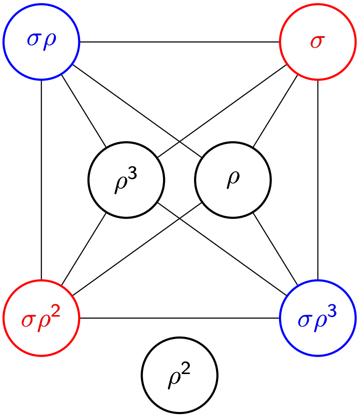

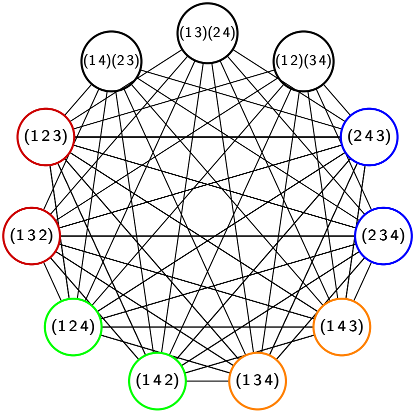

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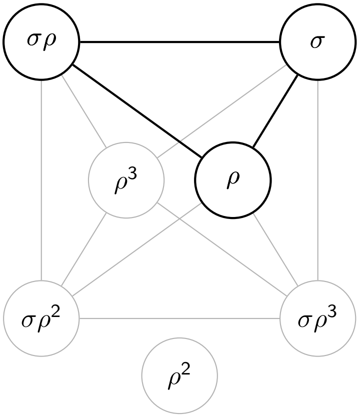

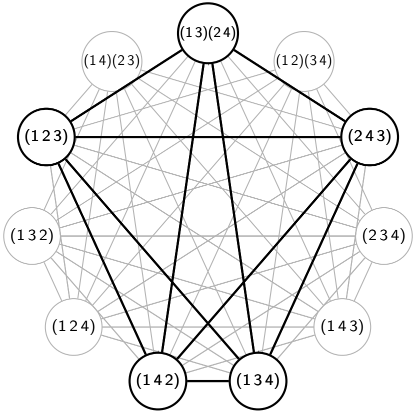

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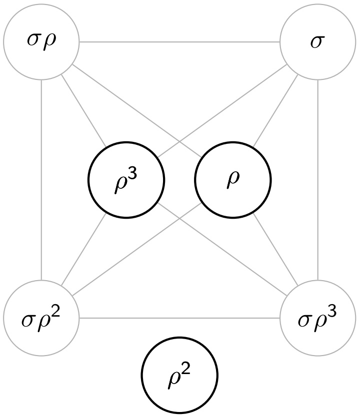

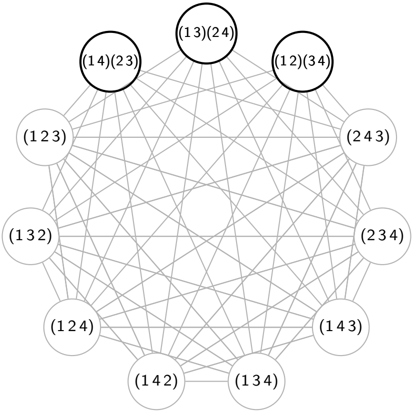

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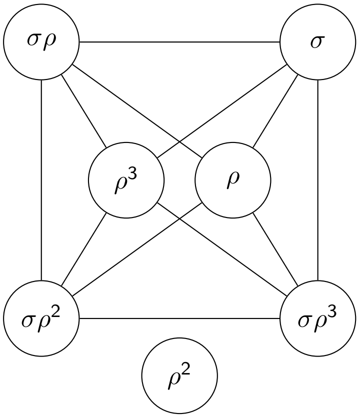

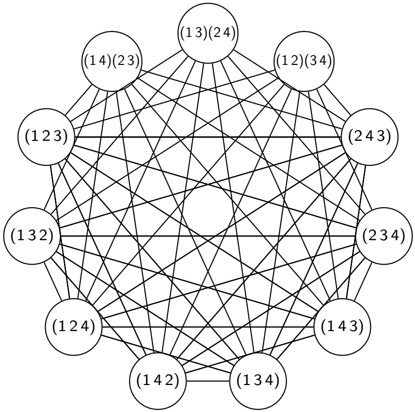

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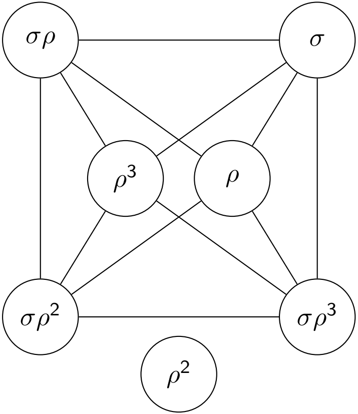

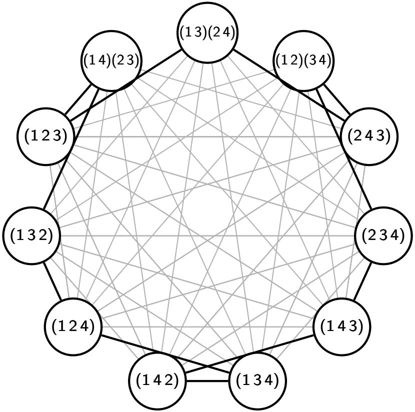

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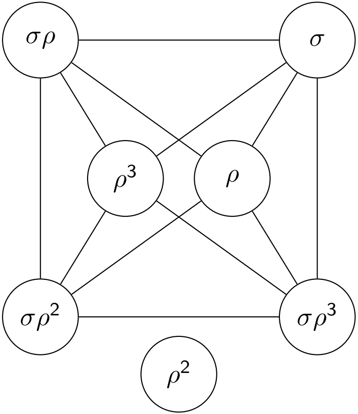

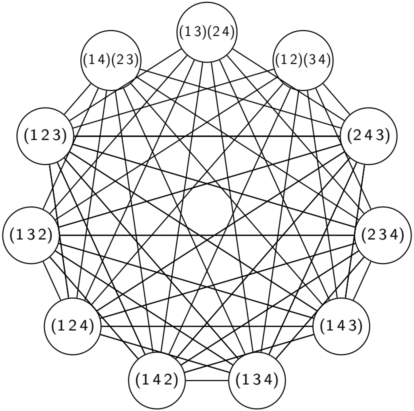

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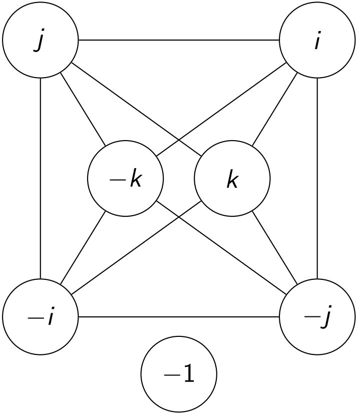

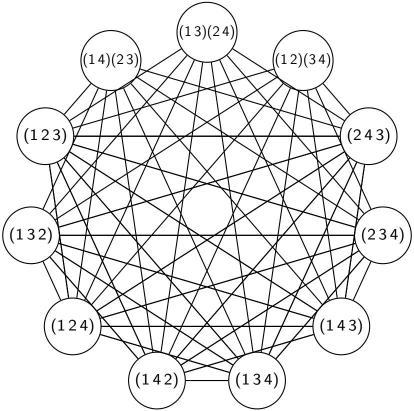

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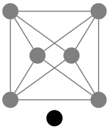
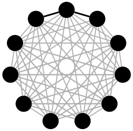
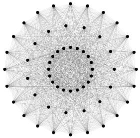
Combinatorial interpretation

Any generating graph either has an isolated vertex or is connected with diameter two.

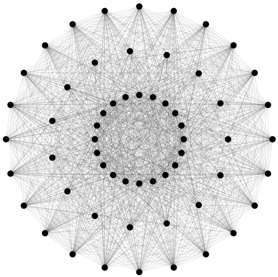






$\frac{3}{2}$-Generation of Finite Groups

Scott Harper



Pure Postgraduate Seminar

10th March 2017