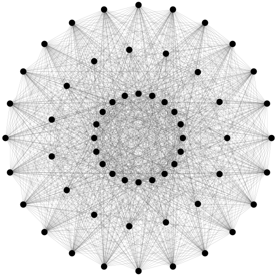
Generating Graphs of Finite Groups

Scott Harper

University of Bristol



Postgraduate Group Theory Conference

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Netto’s Conjecture

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If we arbitrarily select two or more substitutions of n elements, it is to be regarded as extremely probable that the group of lowest order which contains these is the symmetric group, or at least the alternating group. In the case of two substitutions the probability in favor of the symmetric group may be taken as about $\frac{3}{4}$, and in favor of the alternating, but not symmetric, group as about $\frac{1}{4}$. In order that any given substitutions may generate a group which is only a part of the $n!$ possible substitutions, very special relations are necessary, and it is highly improbable that arbitrarily chosen substitutions $s_i = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{i_1} & x_{i_2} & \dots & x_{i_n} \end{pmatrix}$ should satisfy these conditions. The exception most likely to occur would be that all the given substitutions were severally equivalent to an even number of transpositions and would consequently generate the alternating group.

E. Netto, *The theory of substitutions and its application to algebra*, Trans. F. N. Cole, Ann Arbor, Michigan, (1892)

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Theorem (Menezes, Quick & Roney-Dougal, 2013)

If G is simple then $P(G) \geq \frac{53}{90}$ with equality if and only if $G = A_6$.

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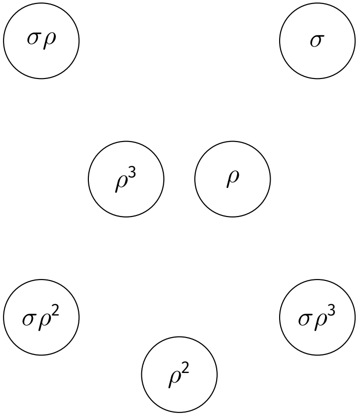

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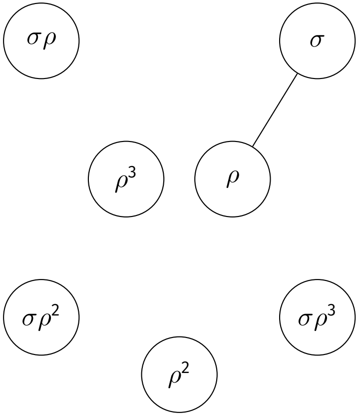

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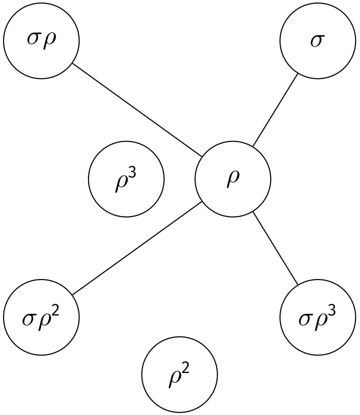

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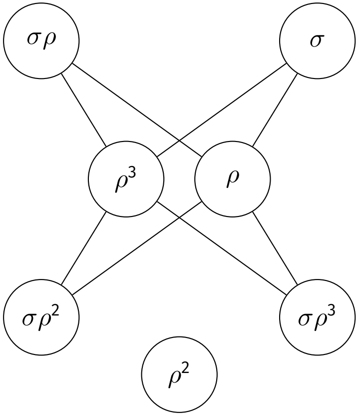

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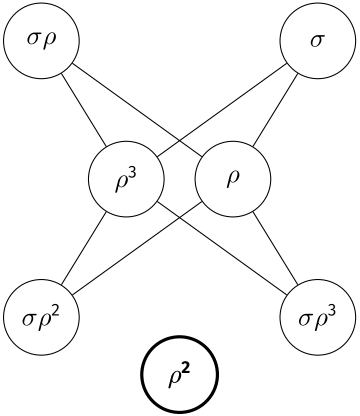

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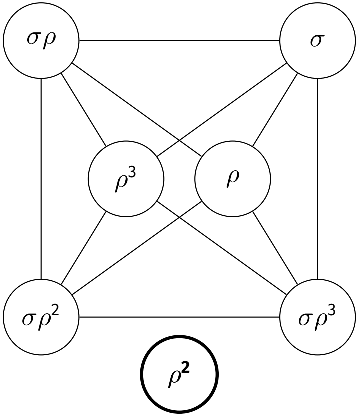

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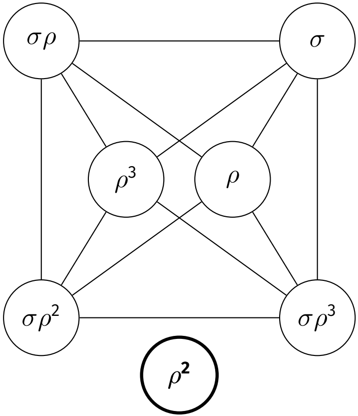

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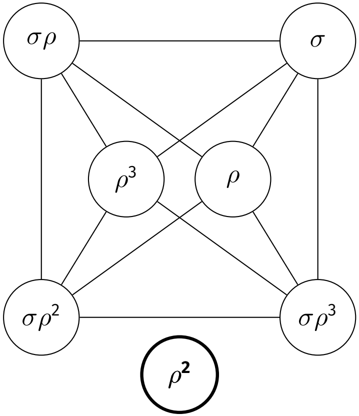

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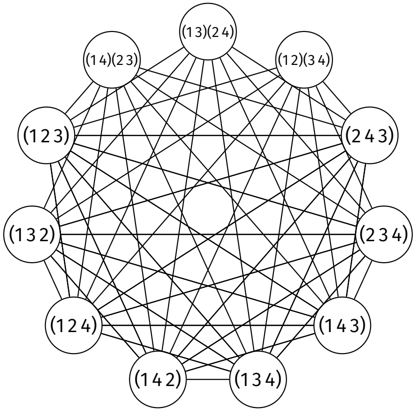

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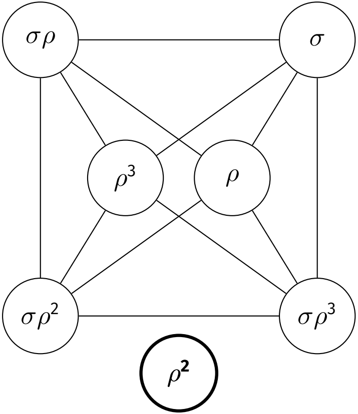

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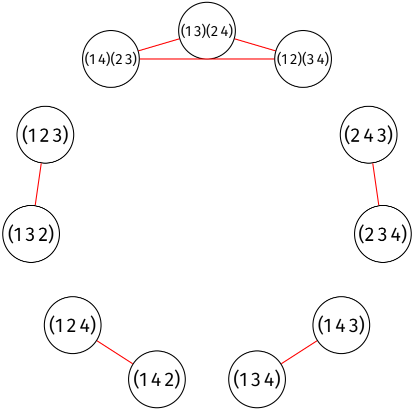

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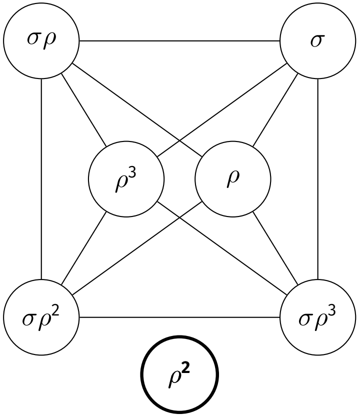

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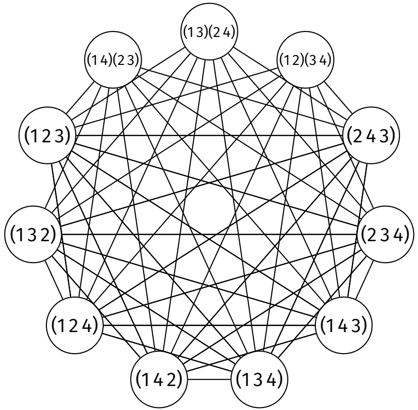

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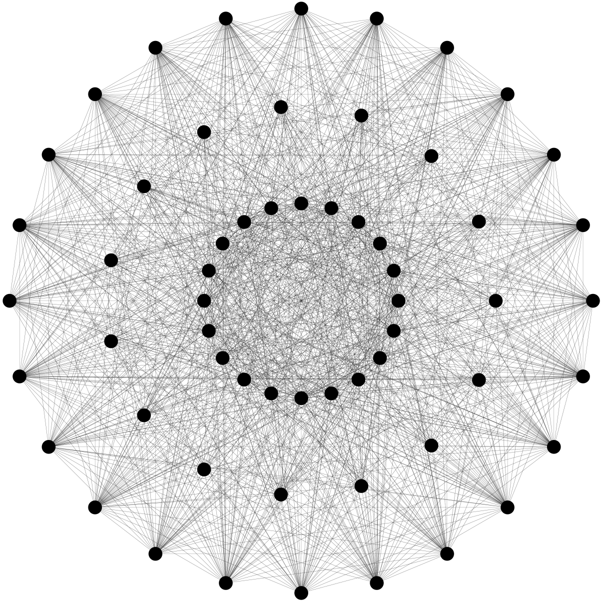

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There exist Tarksi monsters for all $p > 10^{75}$ (Olshanksii, 1979).

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Conjecture (Breuer, Guralnick & Kantor, 2008)

For a finite group G, the generating graph $\Gamma(G)$ has no isolated vertices if and only if every proper quotient of G is cyclic.

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Theorem (Hall, 1936)

Let T be a non-abelian finite simple group. Then T^k is 2-generated if and only if $k \leq \frac{N(T)}{|\mathrm{Aut}(T)|}$ where $N(T)$ is the number of generating pairs of T.

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Theorem (Crestani, Lucchini, 2013)

Let $G = T^k$ for a non-abelian finite simple group T and $k \leq \frac{N(T)}{|\mathrm{Aut}(T)|}$.
Then $\Gamma(G)$ with the isolated vertices removed is connected.

3. Spread

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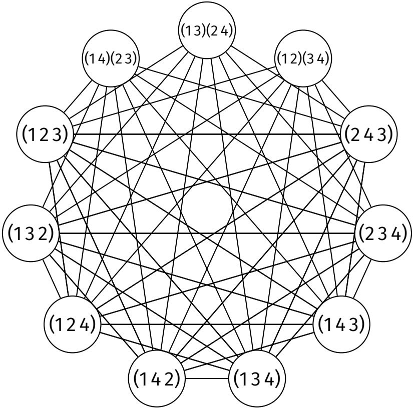

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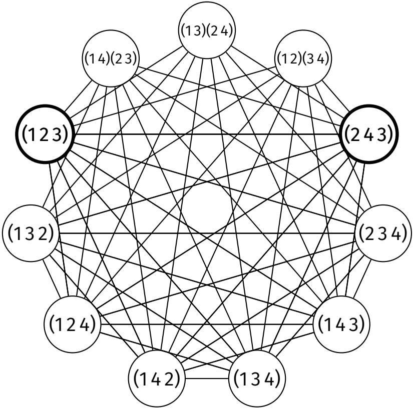

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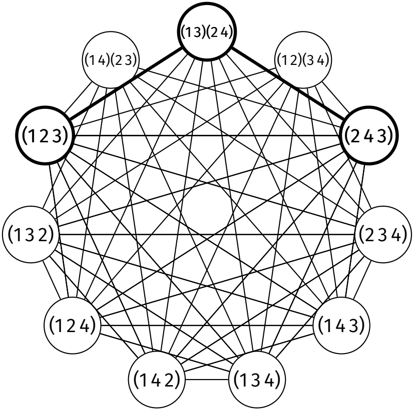

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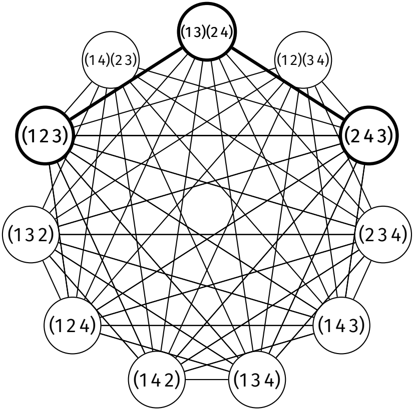
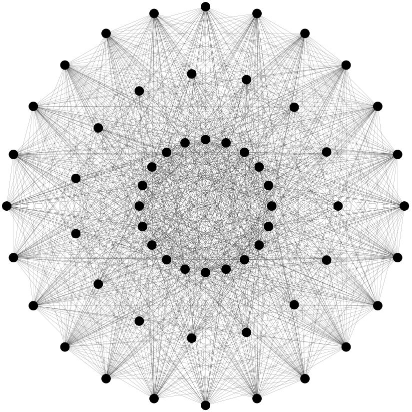


Alternating group A_5



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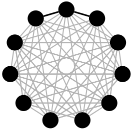

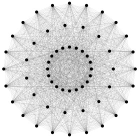

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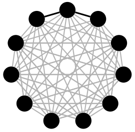

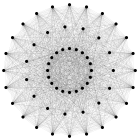

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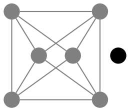

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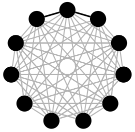

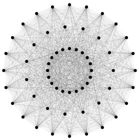

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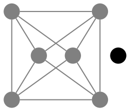

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Theorem (Breuer, Guralnick & Kantor, 2008)

Every finite simple group G has (at least) spread two.

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Main Conjecture

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Examples $G = S_n$ (with $T = A_n$); $\quad G = \mathrm{PGL}_n(q)$ (with $T = \mathrm{PSL}_n(q)$).

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Let T be a simple group of Lie type and let $g \in \mathrm{Aut}(T)$.

Aim: Show that $\Gamma(\langle T, g \rangle)$ has no isolated vertices.

Theorem (H, 2017)

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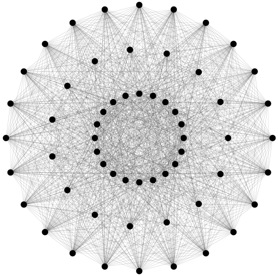
- Bounds on fixed point ratios for almost simple groups

Generating Graphs of Finite Groups

Scott Harper

University of Bristol



Postgraduate Group Theory Conference

27th June 2017