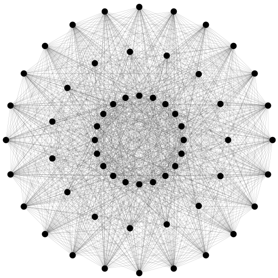
Generating Graphs of Finite Groups

Scott Harper

University of Bristol



Young Algebraists' Conference

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Theorem (CFSG, Steinberg, 1962)

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Netto’s Conjecture

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If we arbitrarily select two or more substitutions of n elements, it is to be regarded as extremely probable that the group of lowest order which contains these is the symmetric group, or at least the alternating group. In the case of two substitutions the probability in favor of the symmetric group may be taken as about $\frac{3}{4}$, and in favor of the alternating, but not symmetric, group as about $\frac{1}{4}$. In order that any given substitutions may generate a group which is only a part of the $n!$ possible substitutions, very special relations are necessary, and it is highly improbable that arbitrarily chosen substitutions $s_i = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ x_{i_1} & x_{i_2} & \ldots & x_{i_n} \end{pmatrix}$ should satisfy these conditions. The exception most likely to occur would be that all the given substitutions were severally equivalent to an even number of transpositions and would consequently generate the alternating group.

E. Netto, *The theory of substitutions and its application to algebra*, Trans. F. N. Cole, Ann Arbor, Michigan, (1892)

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Theorem (Menezes, Quick & Roney-Dougal, 2013)

If G is simple then $P(G) \geq \frac{53}{90}$ with equality if and only if $G = A_6$.

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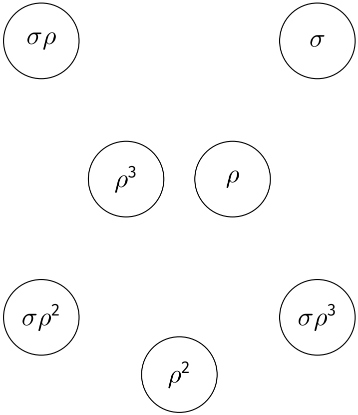

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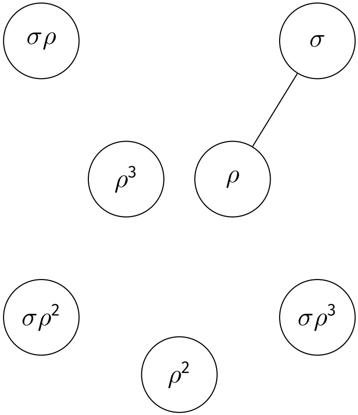

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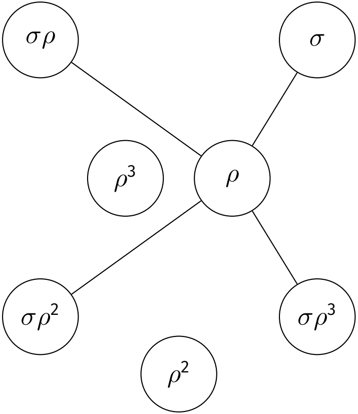

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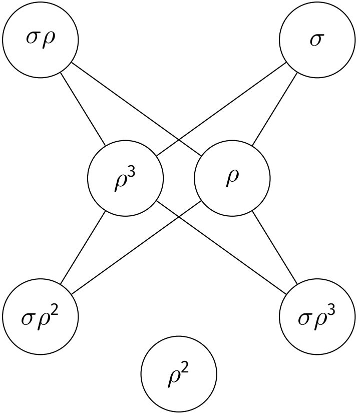

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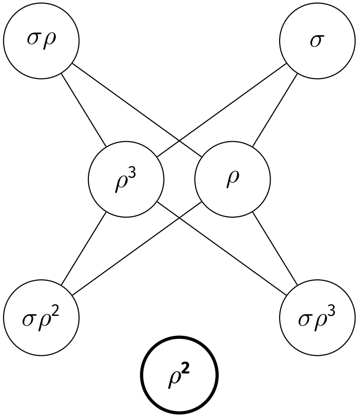

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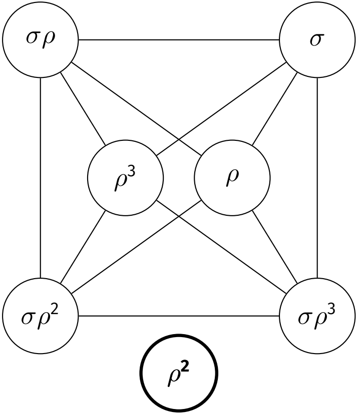

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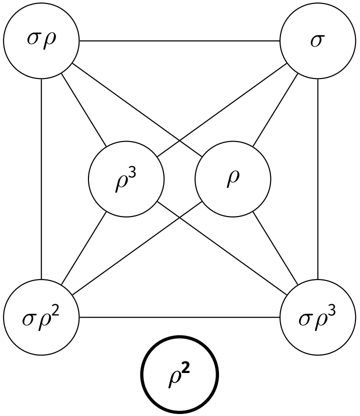

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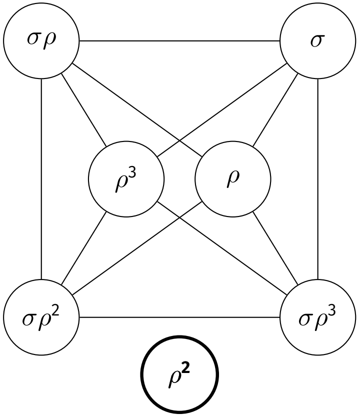

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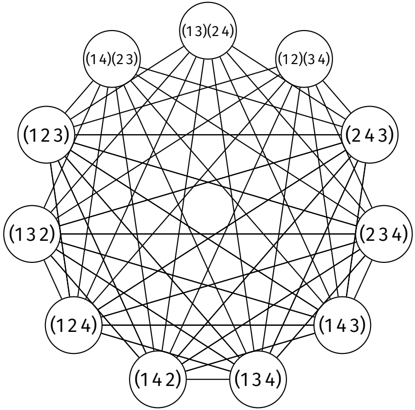

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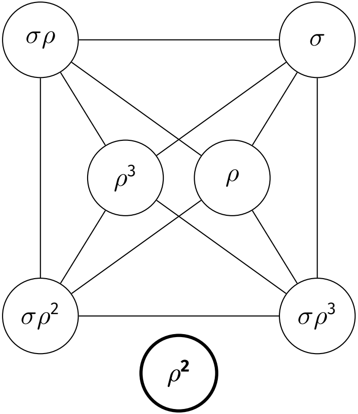

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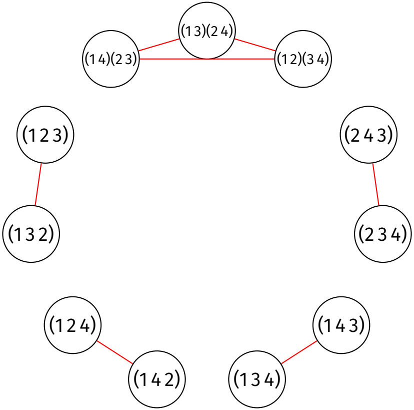

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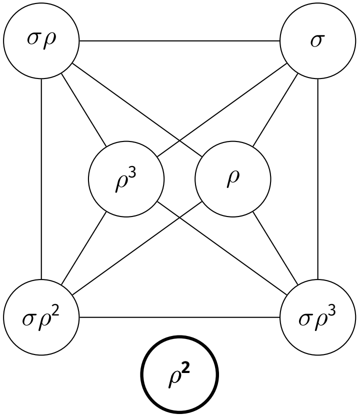

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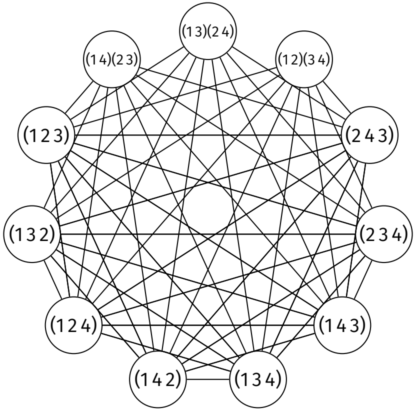

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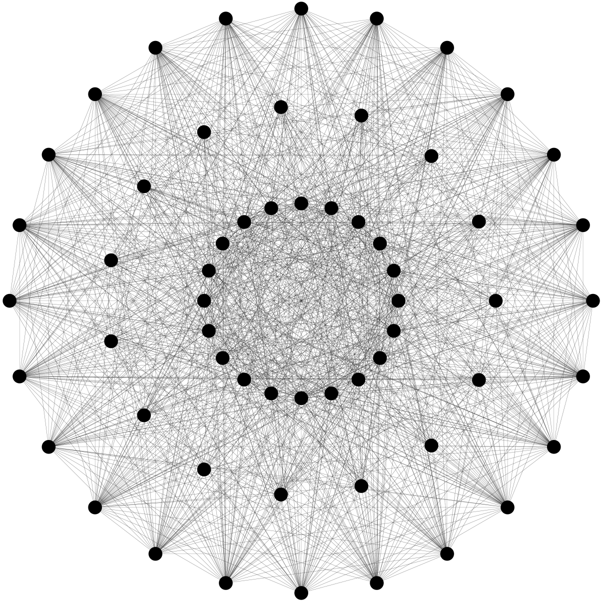

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There exist Tarksi monsters for all $p > 10^{75}$ (Olshanksii, 1979).

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Conjecture (Breuer, Guralnick & Kantor, 2008)

For a finite group G, the generating graph $\Gamma(G)$ has no isolated vertices if and only if every proper quotient of G is cyclic.

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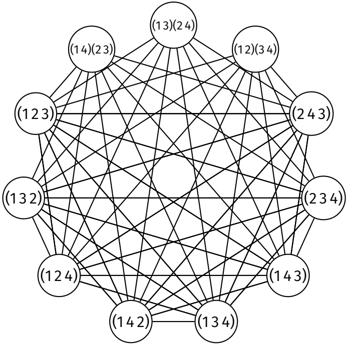

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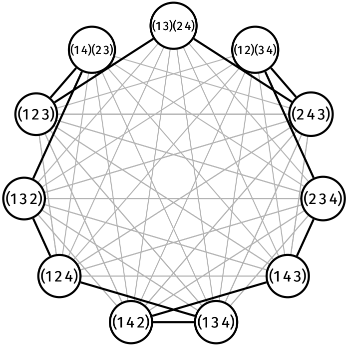

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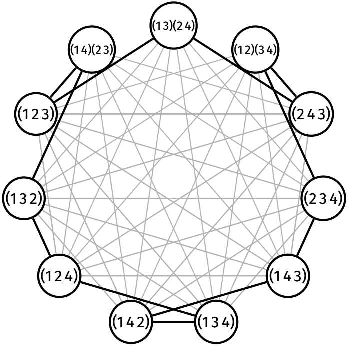

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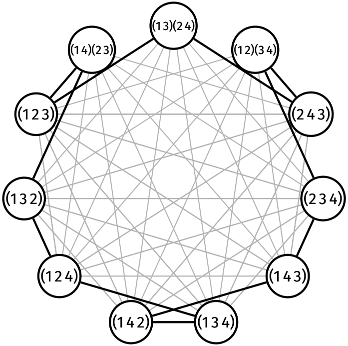

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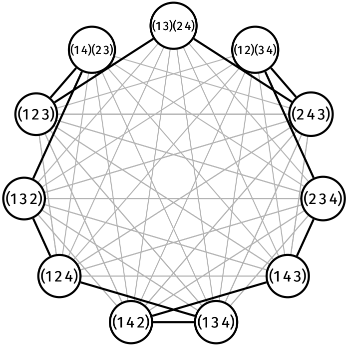

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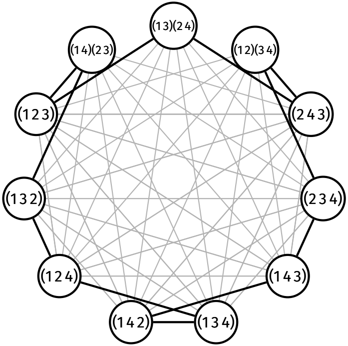

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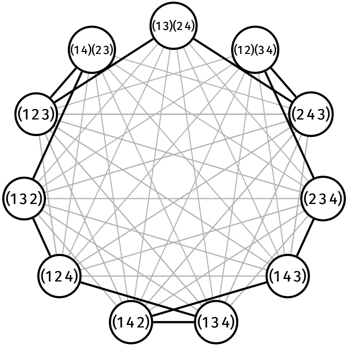

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Conjecture (Breuer, Guralnick, Lucchini, Maroti & Nagy, 2010)

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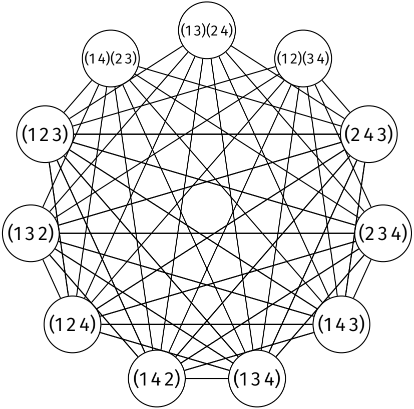

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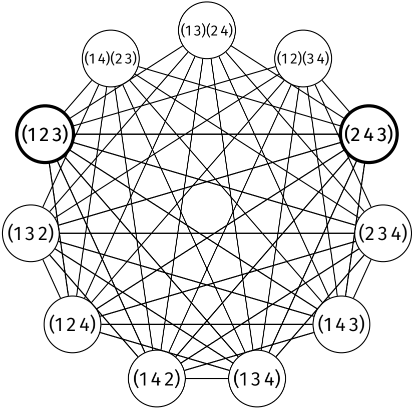

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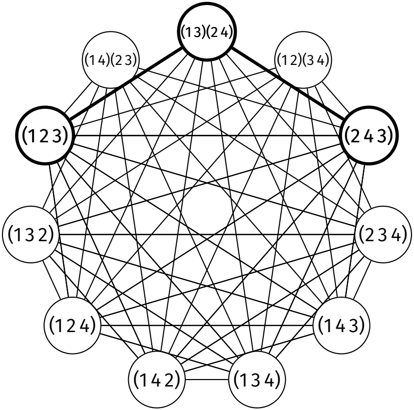

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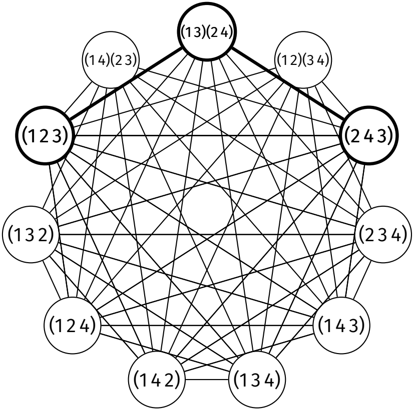

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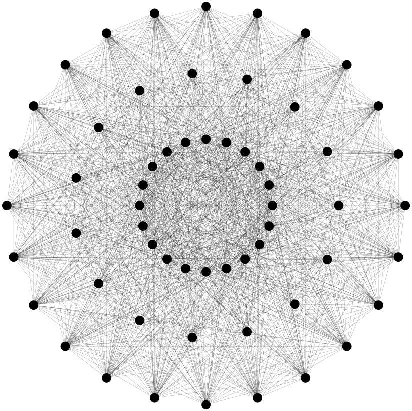

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A group G has spread k if for any distinct $x_1, \ldots, x_k \in G \setminus 1$ there exists an element $z \in G$ such that $\langle x_1, z \rangle = \cdots = \langle x_k, z \rangle = G$.

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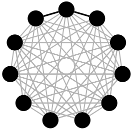
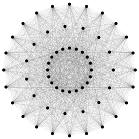
Alternating groups A_4 and A_5





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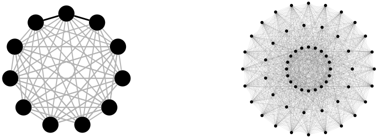

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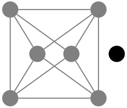
Dihedral group $D_\square$



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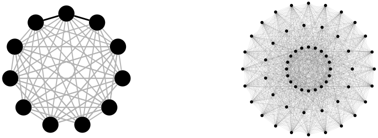

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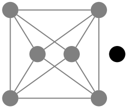

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Theorem (Breuer, Guralnick & Kantor, 2008)

Every finite simple group G has (at least) spread two.

Isolated Vertices

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Examples $G = S_n$ (with $T = A_n$); $G = \mathrm{PGL}_n(q)$ (with $T = \mathrm{PSL}_n(q)$).

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Isolated Vertices and Spread

Let T be a simple group of Lie type and let $g \in \mathrm{Aut}(T)$.

Aim: Show that $\Gamma(\langle T, g \rangle)$ has no isolated vertices.

Theorem (H, 2017)

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Let g be an order two graph-field automorphism of $\mathrm{Sp}_4(q)$.

Then $s(\langle \mathrm{Sp}_4(8), g \rangle) \geq 76$. In general, $s(\langle \mathrm{Sp}_4(q), g \rangle) \geq q^2/18$.

Key Tools

From Simple to Almost Simple

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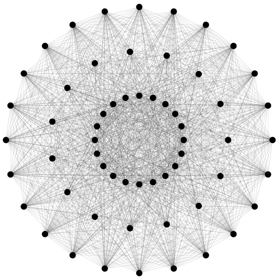
Key Tools

- Shintani descent from the theory of algebraic groups
- Aschbacher's theorem on the maximal subgroups of classical groups
- Bounds on fixed point ratios for almost simple groups

Generating Graphs of Finite Groups

Scott Harper

University of Bristol



Young Algebraists' Conference

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