$\frac{3}{2}$ -Generation of Finite Groups

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Groups St Andrews 7th August 2017

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Question: How are these generating pairs distributed across the group?

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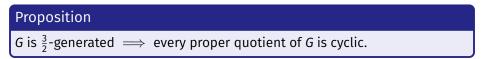
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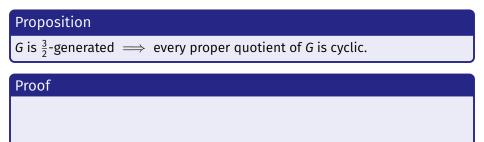
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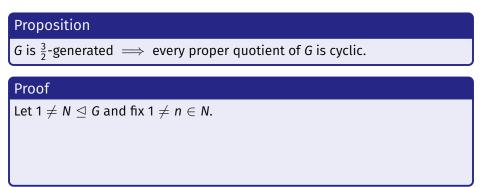
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Which finite groups are $\frac{3}{2}$ -generated?

Simple groups: Groups such that all proper quotients are **trivial**. **Any more?** Groups such that all proper quotients are **cyclic**?







Proposition G is $\frac{3}{2}$ -generated \implies every proper quotient of G is cyclic.

Proof

Let $1 \neq N \trianglelefteq G$ and fix $1 \neq n \in N$. Since G is $\frac{3}{2}$ -generated, there exists $x \in G$ such that $\langle x, n \rangle = G$.

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So G/N is cyclic.

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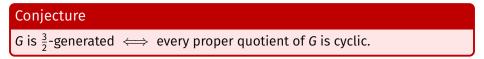
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Conjecture (Breuer, Guralnick & Kantor, 2008)

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Examples $G = S_n$ (with $T = A_n$); $G = PGL_n(q)$ (with $T = PSL_n(q)$).

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Theorem (H, 2017)

If
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A group G has **spread** k if for any $x_1, \ldots, x_k \in G \setminus 1$ there exists an element $z \in G$ such that $\langle x_1, g \rangle = \cdots = \langle x_k, z \rangle = G$.

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Lemma 2
$$P(x,s) \leq \sum_{H \in \mathcal{M}(G,s)} \frac{|x^G \cap H|}{|x^G|}$$

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Question: Which elements in $\text{Sp}_n(q)$ arise as s^e for some $s \notin \text{Sp}_n(q)$?

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Shintani Descent

There is a bijection (with other nice properties)

 $f: X_{\sigma^e}$ -classes of $X_{\sigma^e} \sigma \longrightarrow X_{\sigma}$ -classes of X_{σ}

such that f(g) is X-conjugate to g^e .

Example: $X = \operatorname{Sp}_n(\overline{\mathbb{F}}_2)$ Example: $\sigma : (a_{ij}) \mapsto (a_{ij}^2)$ Example: $X_\sigma = \operatorname{Sp}_n(2)$ Example: $X_{\sigma^e} = \operatorname{Sp}_n(q)$

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Application For all $x \in \text{Sp}_n(2) \leq \text{Sp}_n(q)$ there exists $s \in \text{Sp}_n(q)\sigma$ such that s^e is $\text{Sp}_n(\overline{\mathbb{F}}_2)$ -conjugate to x.

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where A_1 and A_2 act irreducibly on non-degenerate 2- and (n - 2)-spaces.

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Theorem (Aschbacher, 1984)

Let *G* be a classical almost simple group with socle *T*. Any maximal subgroup of *G* which does not contain *T* belongs to one of

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where A_1 and A_2 act irreducibly on non-degenerate 2- and (n - 2)-spaces.

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A power of s^e has an (n-2)-dimensional 1-eigenspace.



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Theorem (Burness, 2007)

Let G be an almost simple classical group, let H be a maximal subgroup of G and let $x \in G$ have prime order. Then

$$|x^G \cap H| < |x^G|^{\varepsilon}$$

for $\varepsilon \approx \frac{1}{2}$, provided that *H* does not stabilise a subspace.

Conjecture

A finite group is $\frac{3}{2}$ -generated iff every proper quotient is cyclic.

Theorem (H, 2017)

If
$$T = \mathsf{PSp}_{2m}(q)$$
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Question: Are there any finite groups with spread one but not two?