$\frac{3}{2}$-Generation of Finite Groups

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Summary: Finite simple groups have many generating pairs.

Question: How are these generating pairs distributed across the group?

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Simple groups: Groups such that all proper quotients are **trivial**.

Any more? Groups such that all proper quotients are **cyclic**?

Let G be a finite group.

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Conjecture (Breuer, Guralnick & Kantor, 2008)

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Examples $G = S_n$ (with $T = A_n$); $G = \mathrm{PGL}_n(q)$ (with $T = \mathrm{PSL}_n(q)$).

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$\frac{3}{2}$-Generation and Spread

Theorem (H, 2017)

If $T = \mathrm{PSp}_{2m}(q)$ or $T = \Omega_{2m+1}(q)$ and $g \in \mathrm{Aut}(T)$, then $\langle T, g \rangle$ is $\frac{3}{2}$-generated.

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A group G has **spread** k if for any $x_1, \dots, x_k \in G \setminus 1$ there exists an element $z \in G$ such that $\langle x_1, g \rangle = \cdots = \langle x_k, z \rangle = G$.

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If G is a finite simple group, then $s(G) \geq 2$.

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Lemma 2

$$P(x,s) \leq \sum_{H \in \mathcal{M}(G,s)} \frac{|x^G \cap H|}{|x^G|}$$

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Question: Which elements in $\mathrm{Sp}_n(q)$ arise as s^e for some $s \notin \mathrm{Sp}_n(q)$?

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Application For all $x \in \mathrm{Sp}_n(2) \leq \mathrm{Sp}_n(q)$ there exists $s \in \mathrm{Sp}_n(q)\sigma$ such that s^e is $\mathrm{Sp}_n(\overline{\mathbb{F}}_2)$-conjugate to x.

Choose $s \in \mathrm{Sp}_n(q)\sigma$ such that s^e has the form

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Theorem (Burness, 2007)

Let G be an almost simple classical group, let H be a maximal subgroup of G and let $x \in G$ have prime order. Then

$$|x^G \cap H| < |x^G|^{\varepsilon}$$

for $\varepsilon \approx \frac{1}{2}$, provided that H does not stabilise a subspace.

Summary

Conjecture

A finite group is $\frac{3}{2}$-generated iff every proper quotient is cyclic.

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If $T = \mathrm{PSp}_{2m}(q)$ or $T = \Omega_{2m+1}(q)$ and $g \in \mathrm{Aut}(T)$, then $s(\langle T, g \rangle) \geq 2$.

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Question: Are there any finite groups with spread one but not two?