# $\frac{3}{2}$ -Generation of Finite Groups

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Groups St Andrews 7th August 2017

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Question: How are these generating pairs distributed across the group?

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#### Simple groups: Groups such that all proper quotients are trivial.

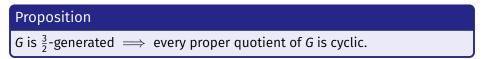
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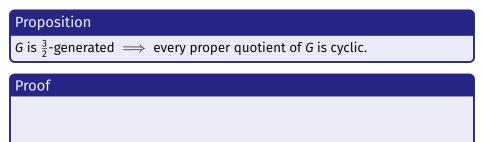
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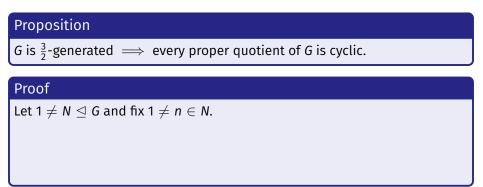
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**Simple groups:** Groups such that all proper quotients are **trivial**. **Any more?** Groups such that all proper quotients are **cyclic**?







# Proposition G is $\frac{3}{2}$ -generated $\implies$ every proper quotient of G is cyclic.

#### Proof

Let  $1 \neq N \trianglelefteq G$  and fix  $1 \neq n \in N$ . Since G is  $\frac{3}{2}$ -generated, there exists  $x \in G$  such that  $\langle x, n \rangle = G$ .

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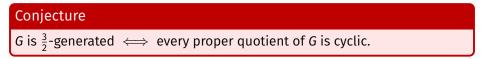
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#### Conjecture (Breuer, Guralnick & Kantor, 2008)

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It suffices to show: For all finite almost simple groups G,

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**Examples**  $G = S_n$  (with  $T = A_n$ );  $G = PGL_n(q)$  (with  $T = PSL_n(q)$ ).

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### Theorem (H, 2017)

If 
$$T = \mathsf{PSp}_{2m}(q)$$
 or  $T = \Omega_{2m+1}(q)$  and  $g \in \mathsf{Aut}(T)$ , then  $\langle T, g \rangle$  is  $\frac{3}{2}$ -generated.

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A group G has **spread** k if for any  $x_1, \ldots, x_k \in G \setminus 1$  there exists an element  $z \in G$  such that  $\langle x_1, g \rangle = \cdots = \langle x_k, z \rangle = G$ .

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If G is a finite simple group, then  $s(G) \ge 2$ .

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Lemma 2
$$P(x,s) \leq \sum_{H \in \mathcal{M}(G,s)} \frac{|x^G \cap H|}{|x^G|}$$

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**Question:** Which elements in  $\text{Sp}_n(q)$  arise as  $s^e$  for some  $s \notin \text{Sp}_n(q)$ ?

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There is a bijection (with other nice properties)

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**Application** For all  $x \in \text{Sp}_n(2) \leq \text{Sp}_n(q)$  there exists  $s \in \text{Sp}_n(q)\sigma$  such that  $s^e$  is  $\text{Sp}_n(\overline{\mathbb{F}}_2)$ -conjugate to x.

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**Key Features** Only two subspaces are stabilised by *s*<sup>*e*</sup>.

A power of  $s^e$  has an (n-2)-dimensional 1-eigenspace.



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### Theorem (Burness, 2007)

Let G be an almost simple classical group, let H be a maximal subgroup of G and let  $x \in G$  have prime order. Then

$$|x^G \cap H| < |x^G|^{\varepsilon}$$

for  $\varepsilon \approx \frac{1}{2}$ , provided that *H* does not stabilise a subspace.

# Conjecture

A finite group is  $\frac{3}{2}$ -generated iff every proper quotient is cyclic.

# Theorem (H, 2017)

If 
$$T = \mathsf{PSp}_{2m}(q)$$
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Question: Are there any finite groups with spread one but not two?