Uniform Domination in Simple Groups

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(joint with Tim Burness)

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Groups and Geometry

University of Auckland

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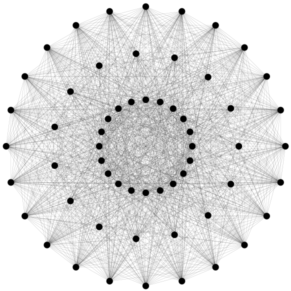

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Question: How are the generating pairs distributed across the group?

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The **uniform domination number** of G is the minimal size of a set S of conjugate elements of G such that for each non-identity element $x \in G$ there exists $s \in S$ such that $\langle x, s \rangle = G$.

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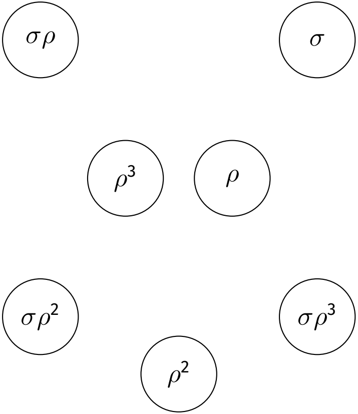

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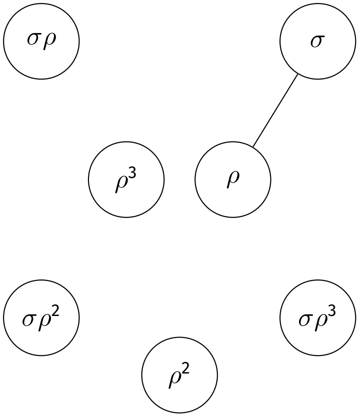

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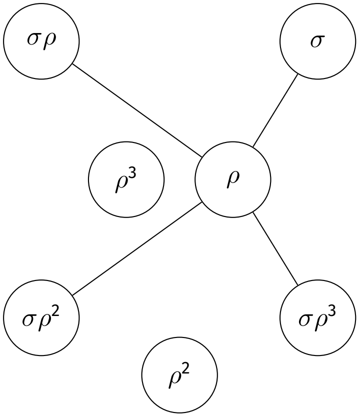

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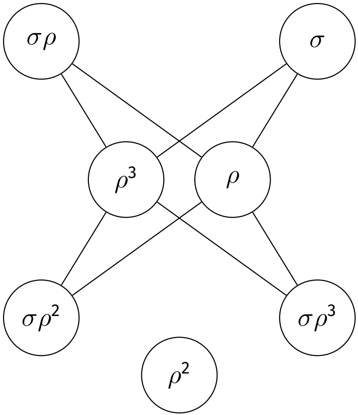

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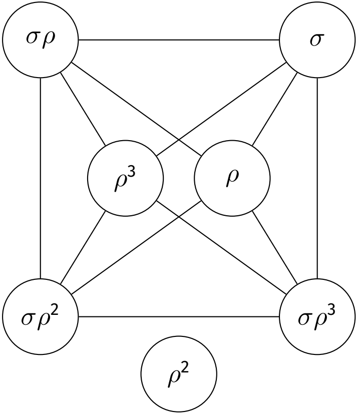

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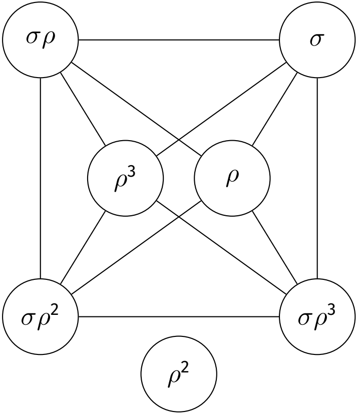

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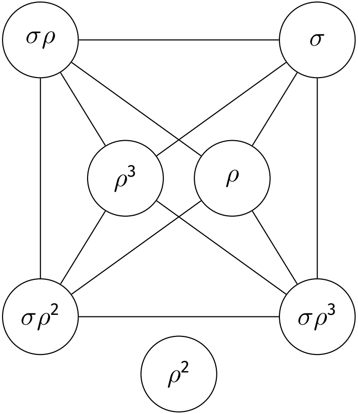

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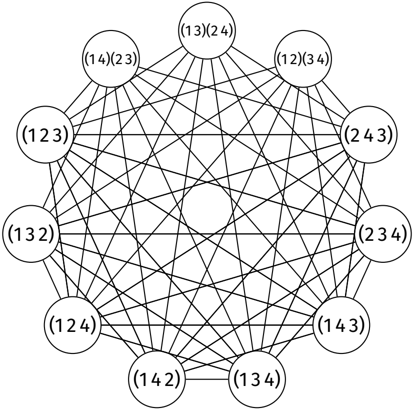

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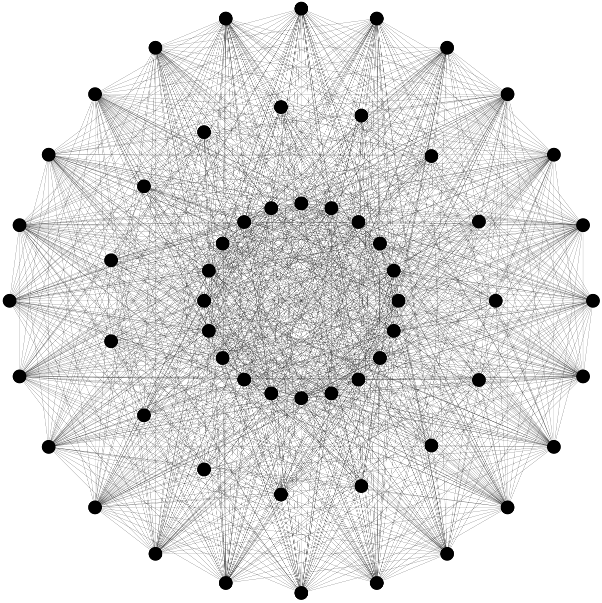

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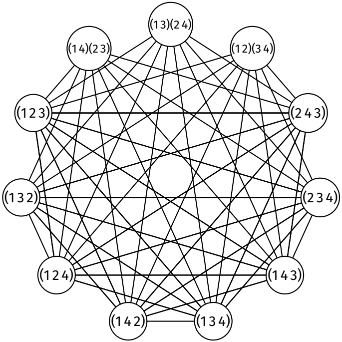

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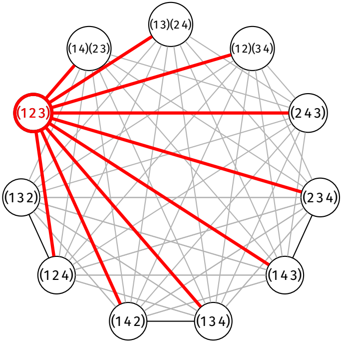

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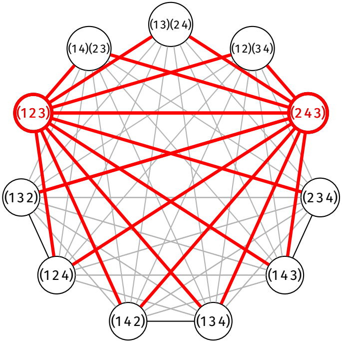

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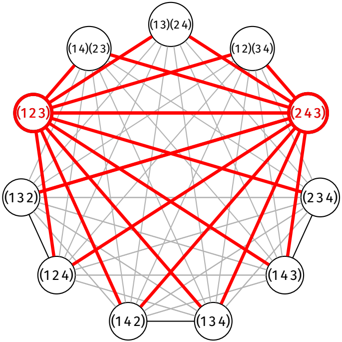

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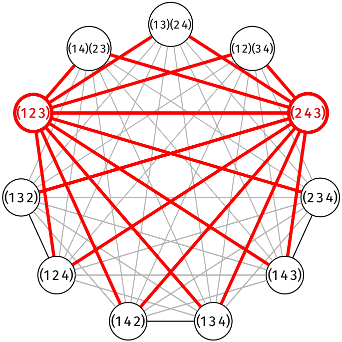
$\implies \gamma_t(\Gamma(A_4)) = 2$



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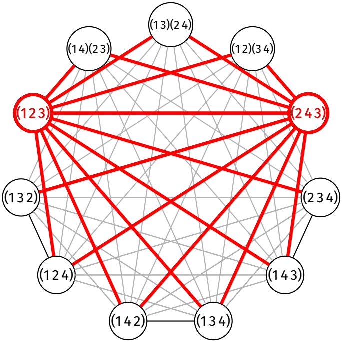

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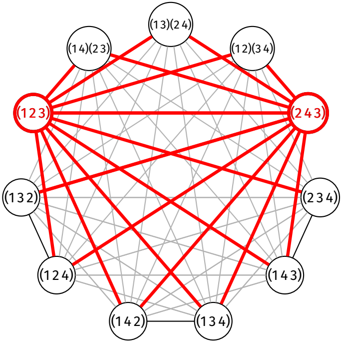

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By the Classification of Finite Simple Groups we need to consider

- alternating groups ($A_5, A_6, A_7, \dots$)
- classical groups (e.g. $\mathrm{PSL}_n(q), \mathrm{P}\Omega^-_{2m}(q), \dots$)
- exceptional groups (e.g. $E_8(q), {}^2B_2(q), \dots$)
- sporadic groups (e.g. $\mathrm{M}_{24}, \mathbb{M}, \dots$)

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Observation: If $H \leqslant G$ is core-free, then G acts faithfully on G/H and $\{Hg_1, \ldots, Hg_c\}$ is a base iff $\bigcap_{i=1}^{c} H^{g_i} = 1$.

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Therefore, $b(G, G/H) \leqslant 6$, so by the Bases Lemma $\gamma_u(G) \leqslant 6$. ■

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"Probabilistic Lemma"

$$Q(G, s, c) \leqslant \sum_{i=1}^{k} |x_i^G| \left(\sum_{H \in \mathcal{M}(G,s)} \mathrm{fpr}(x_i, G/H) \right)^c$$

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Let $x_1, \ldots, x_k$ be prime order conjugacy classes representatives in G.

"Probabilistic Lemma"

$$Q(G, s, c) \leqslant \sum_{i=1}^{k} |x_i^G| \left(\sum_{H \in \mathcal{M}(G,s)} \mathrm{fpr}(x_i, G/H) \right)^c$$

If $\mathcal{M}(G, s) = \{H\}$, then the Probabilistic Lemma is the probabilistic approach introduced by Liebeck and Shalev for base sizes.

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By the Probabilistic Lemma, $\gamma_u(G) \leqslant 2r + 26$. ■

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- If G is an **exceptional** group, then $\gamma_u(G) \leqslant 6$.
- If G is a **sporadic** group, then $\gamma_u(G) \leqslant 4$.