The Distinguishing Number of Semiprimitive Groups

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Symmetry Breaking in Discrete Structures

Casa Matemática, Oaxaca

20th September 2018

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Background

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Question When is $D(G) = 2$?

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Example 4. Primitive groups

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Question Can we extend this result to a wider class of groups?

The **base size** of G, written ***b*(*G*)**, is the least $k \geqslant 1$ for which there exists a subset $B \subseteq \Omega$ of size k such that

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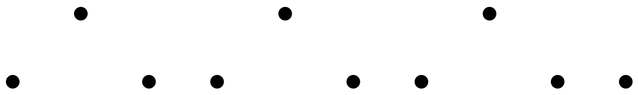

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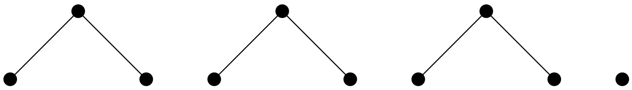

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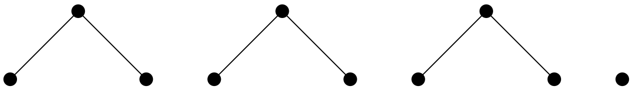

Observation $D(G) \leqslant b(G) + 1$

Theorem (Halasi, Liebeck, Maróti '18)
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Results

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> **Theorem 2** (Devillers, H., Morgan '18)
> Let G be a nonquasiprimitive semiprimitive group. Then $D(G) = 2$ or G is $\mathrm{GL}_2(3)$ on the nonzero vectors of $\mathbb{F}_2^3$ and $D(G) = 3$.

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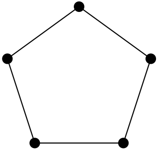
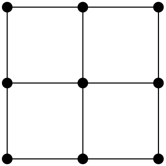


C_5



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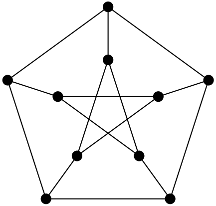
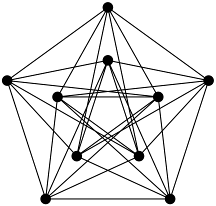


$\mathbb{P}$



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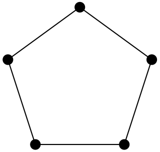
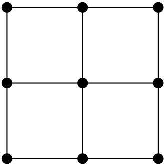


C_5



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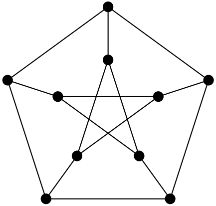

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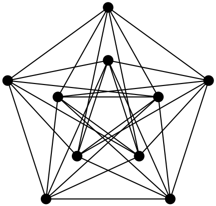

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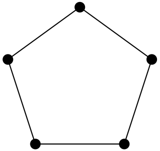
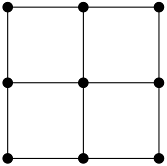


C_5



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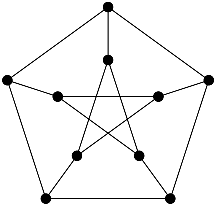

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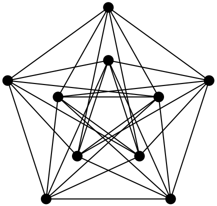

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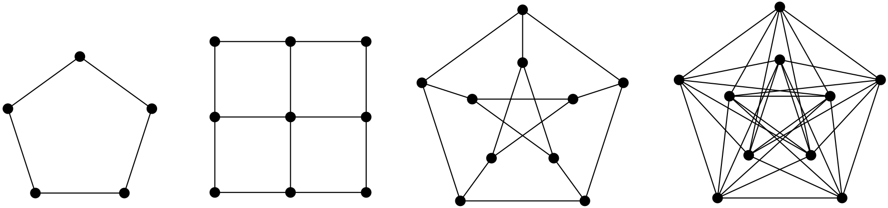

C_5 $\quad K_3 \,\square\, K_3$ $\quad \mathbb{P}$ $\quad \overline{\mathbb{P}}$

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If Γ is 2-arc-transitive and not bipartite, then $\mathrm{Aut}(\Gamma)$ is semiprimitive.

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Example 3. Simple groups

Let $G \leqslant \mathrm{Sym}(\Omega)$ be simple and transitive.

Then G is quasiprimitive, so $D(G) = 2$ or $G = \mathrm{Alt}(\Omega)$ or G is one of the 13 simple primitive groups in $\mathcal{P}$.

Methods

Investigate the action of G on Ω by studying the induced actions of G on G-invariant partitions of Ω.

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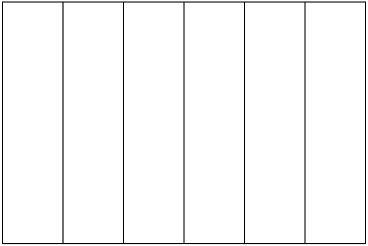

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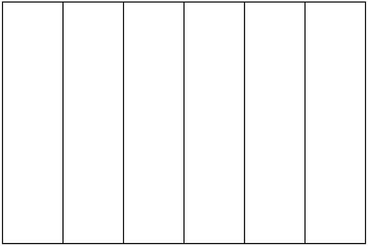

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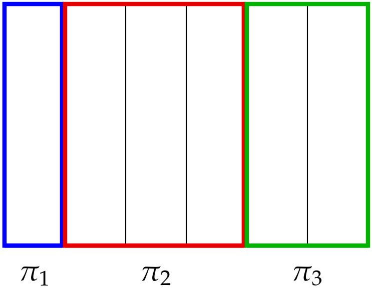

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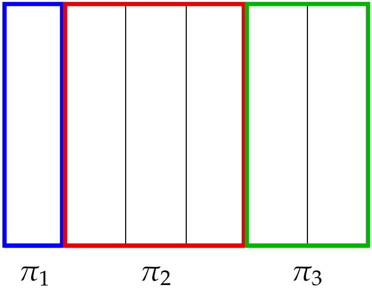

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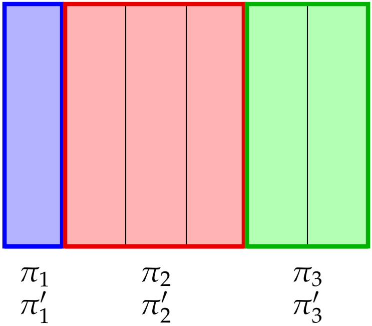

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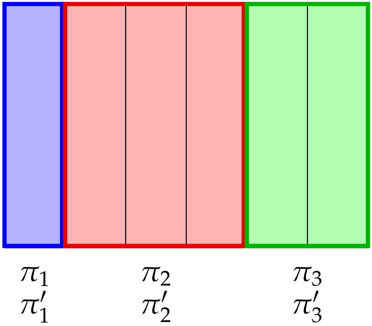

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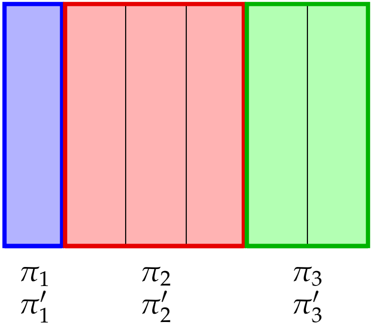

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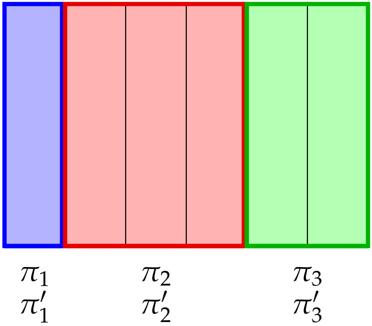

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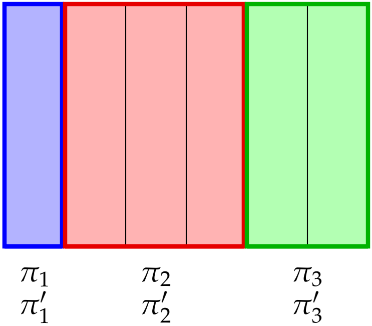

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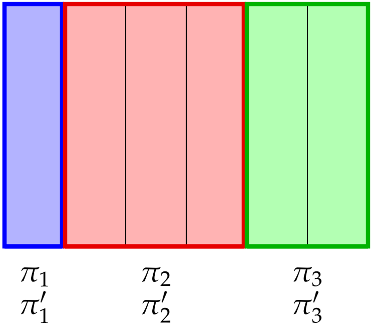

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Therefore, $g = 1$, so Π' witnesses $D(G) \leqslant d = D(G^\Sigma)$.

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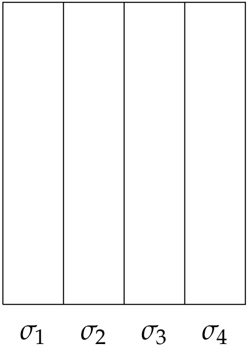

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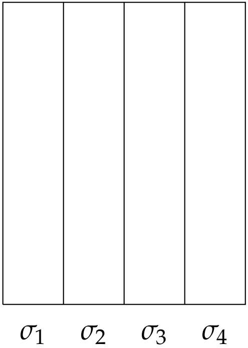

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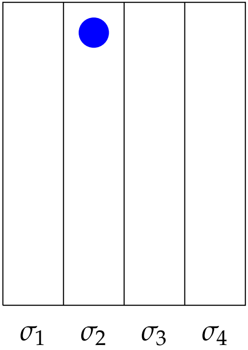

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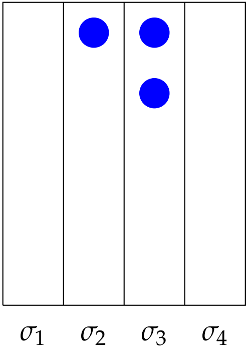

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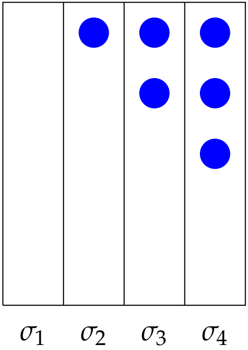

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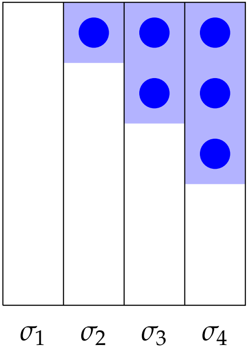

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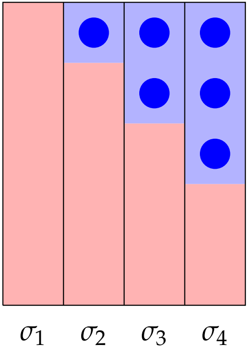

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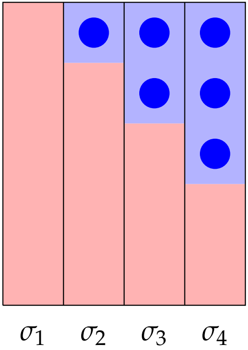

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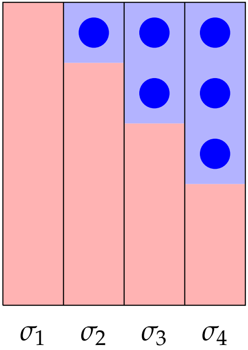

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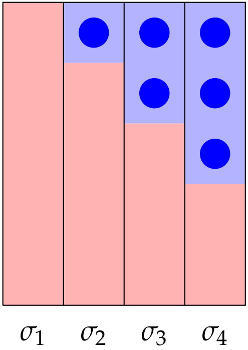

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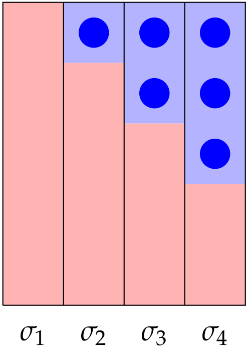

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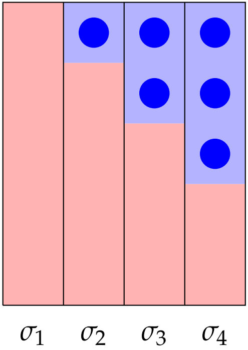

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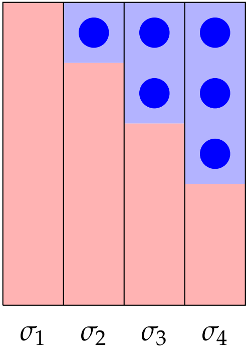

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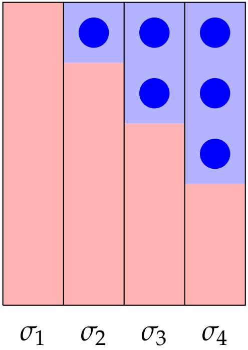

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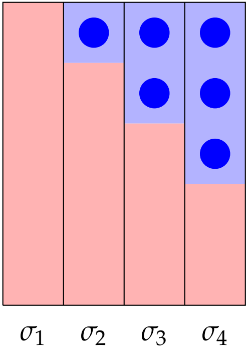

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Sketch of Proof

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