What is a group?

Scott Harper

Postgraduate Seminar

2nd November 2018

What is a group?

It depends who you ask.

Groups are viewed from many different perspectives by a diversity of people with a wide range of differing interests. Sometimes two seemingly unrelated properties of a group, coming from two radically distinct viewpoints, miraculously happen to be equivalent. I'll tell you about some of the different perspectives on group theory and then focus on one case where a surprising bridge was built, the Muller–Schupp theorem.

I expect the audience to arrive knowing what a group is and leave doubting that they ever did.

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"depends on who you ask" vs "depends on whom you ask" [duplicate]

1

2

This question already has an answer here:
How can one differentiate between "who" and "whom"? *5 answers*

Use the adjective *exalted* to describe something (or someone) that is raised in rank, value, or power.

> *Which group has the most exalted status at your high school depends on* ***who*** *you ask.*

Is the sentence correct or should we use *whom* instead of *who*?

pronouns case who-whom

share improve this question

edited Jan 21 '17 at 22:11
Mari-Lou A
13.2k 7 38 75

asked Nov 8 '16 at 7:03
Ben
137 2 8

marked as duplicate by Catija ♦, snailboat ♦ Dec 13 '16 at 13:56

This question has been asked before and already has an answer. If those answers do not fully address your question, please ask a new question.

asked 1 year, 11 months ago

viewed 7,446 times

active 1 year, 8 months ago

Linked

- 33 How can one differentiate between "who" and "whom"?

Related

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- 0 "Who/whom was this street named after?"
- 5 Who or whom? "Figure 1 depicts a surveillance system, detecting pedestrians whom are crossing dangerous regions."
- 1 Who has created this group? Who have created this group?
- 3 Difference between "do it yourself" and "do it on your own"

> *Which group has the most exalted status at your high school depends on **who** you ask.*

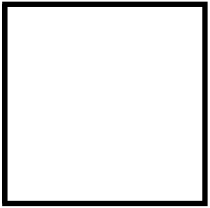
D_8

D_8

symmetries of a square

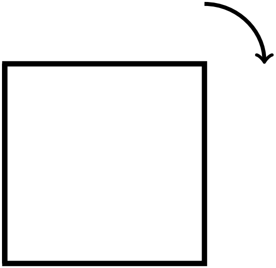
D_8

symmetries of a square



D_8

symmetries of a square



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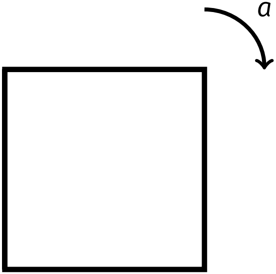

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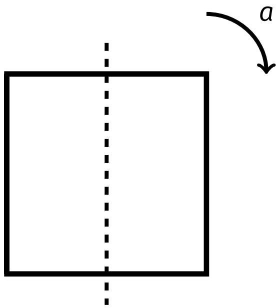

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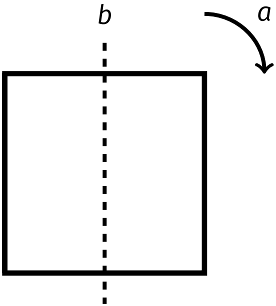

Abstract groups

A **group** is a set together with a binary operation satisfying …

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1	1	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	1	ab	a^2b	a^3b	b
a^2	a^2	a^3	1	a	a^2b	a^3b	b	a^2b
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	ab^3	a^2b	ab	1	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	1	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	1	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	1

D_8

Abstract group

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a^2	a^2	a^3	1	a	a^2b	a^3b	b	a^2b
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	ab^3	a^2b	ab	1	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	1	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	1	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	1

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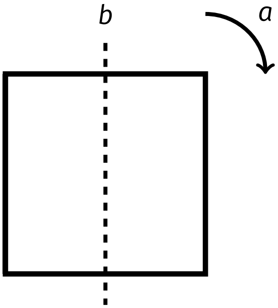
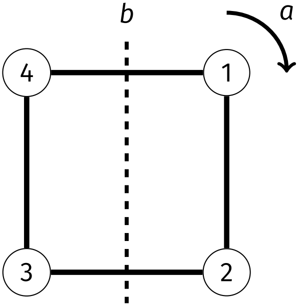


Permutation groups



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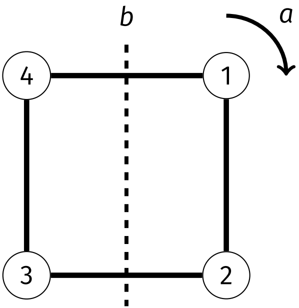
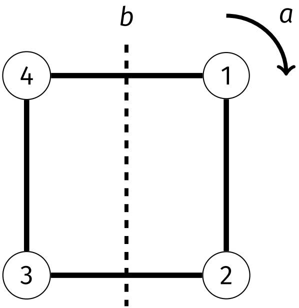


$a = (1\ 2\ 3\ 4) \quad b = (1\ 4)(2\ 3)$

Permutation groups



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$$D_8 = \langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle \leqslant \mathrm{Sym}(4)$$

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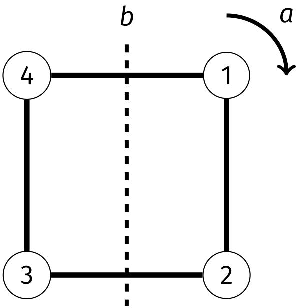

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A **permutation group** is a subgroup $G \leqslant \mathrm{Sym}(\Omega)$.

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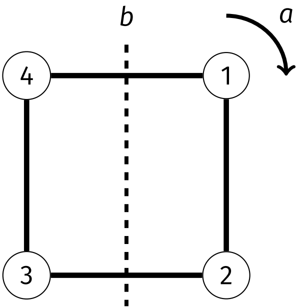

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Cayley's Theorem Every group is a permutation group.

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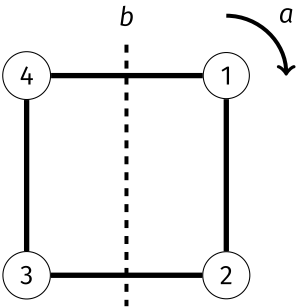

symmetries of an oblong

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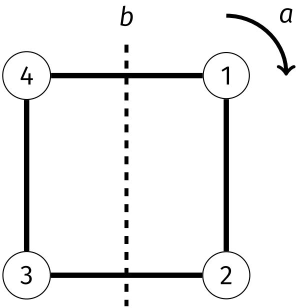

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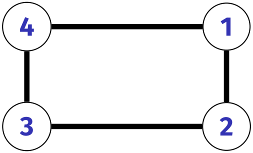

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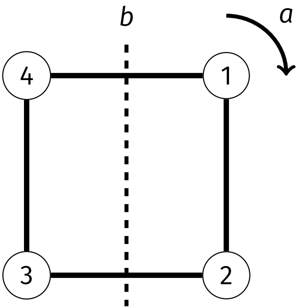

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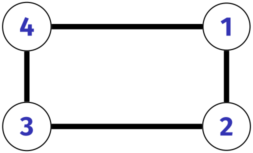

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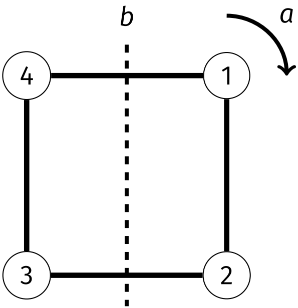

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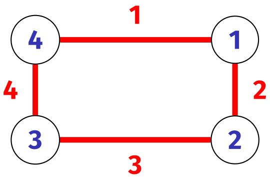

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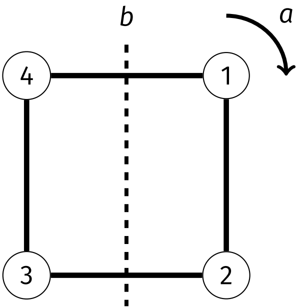

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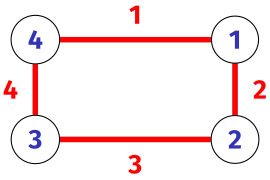

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PREFACE TO THE FIRST EDITION

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; [redacted] My answer to this question is that [redacted] in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, [redacted]

W. BURNSIDE

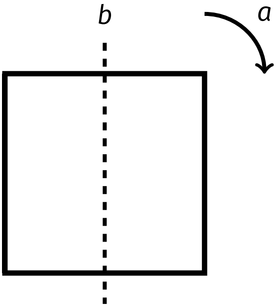
July 1897

D_8

Permutation group

$\langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle$

symmetries of a square



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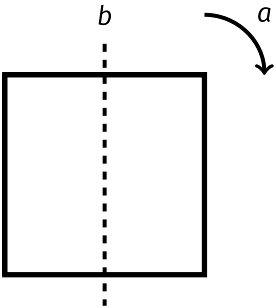
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	ab^3	a^2b	ab	1	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	1	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	1	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	1

Matrix groups



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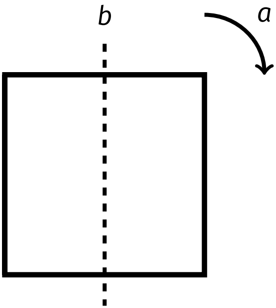

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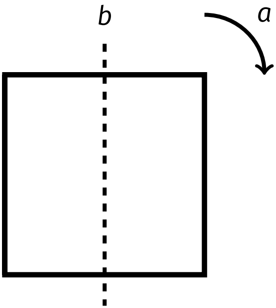

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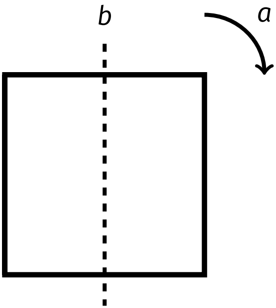

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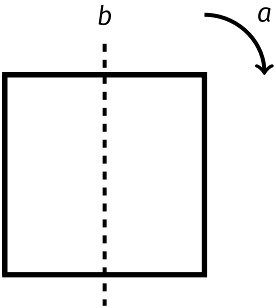
A **representation** is a homomorphism $\varphi\colon G \to \mathrm{GL}_d(F)$.

Representation theory



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Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

W. BURNSIDE

July 1897

Theorem (Burnside, 1904)

Let p and q be prime. Any group of order $p^a q^b$ is soluble.

William Burnside

Theorem (Burnside, 1904)
Let p and q be prime. Any group of order $p^a q^b$ is soluble.

Ambrose Burnside

PREFACE TO THE SECOND EDITION

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

W. BURNSIDE

March 1911

D_8

Permutation group

$$\langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle$$

symmetries of a square

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a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	ab^3	a^2b	ab	1	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	1	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	1	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	1

Representation theory

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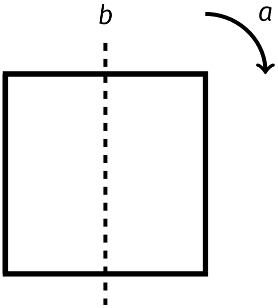
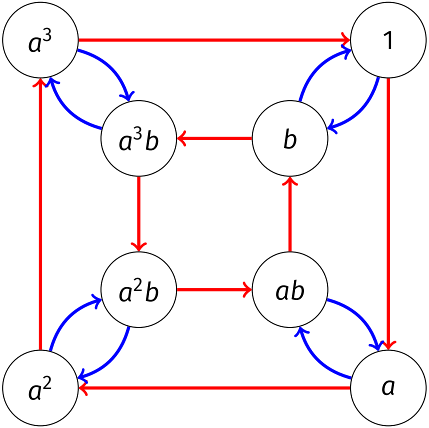
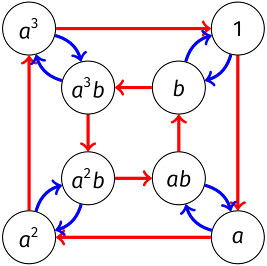


Geometric group theory

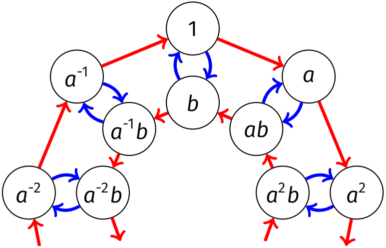


Cayley graph of D_8 with respect to $\{a, b\}$

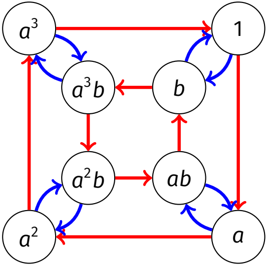
$$\Gamma(D_8, \{a, b\})$$



$\Gamma(D_8, \{a, b\})$



$\Gamma(D_\infty, \{a, b\})$



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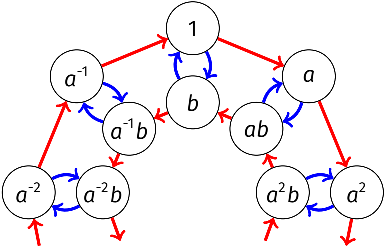

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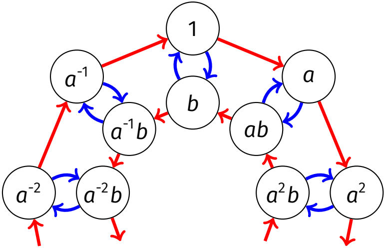

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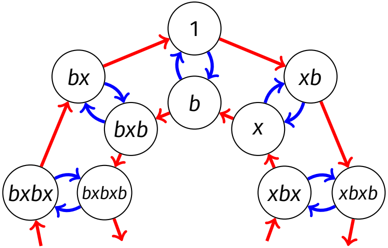

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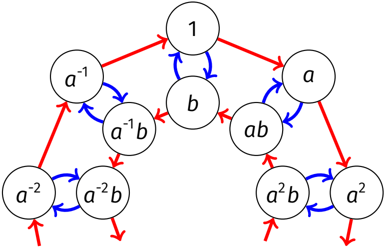

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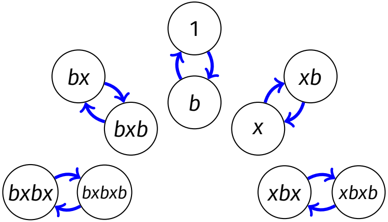
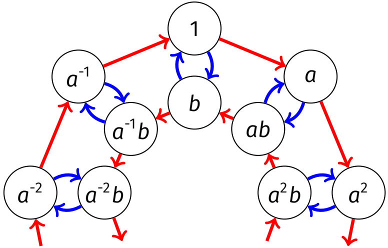


$\Gamma(D_\infty, \{\quad, b\})$



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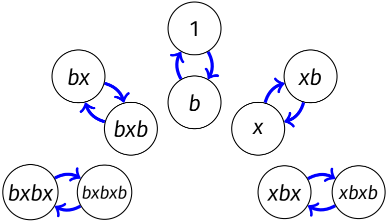

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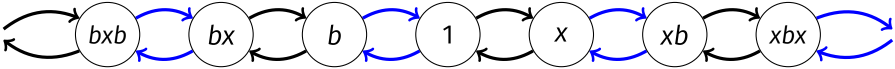
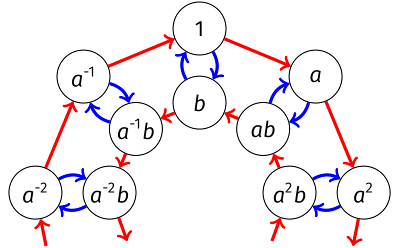


$\Gamma(D_\infty, \{x, b\})$



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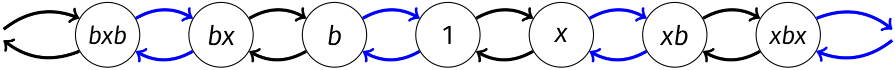

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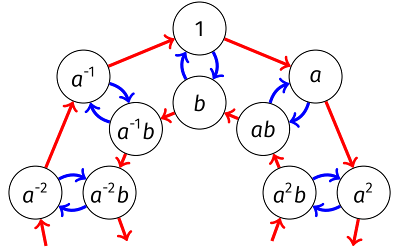
Different generating sets give different Cayley graphs



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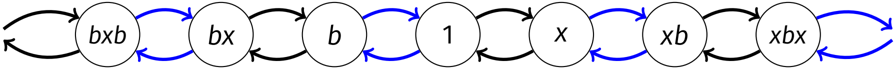

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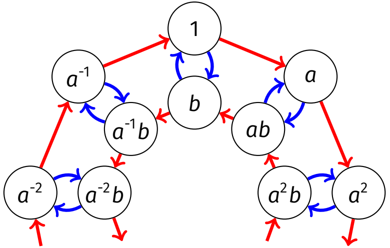
Different generating sets give different Cayley graphs

– Think up to **quasiisometry**



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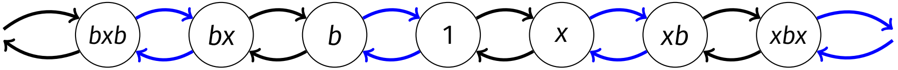

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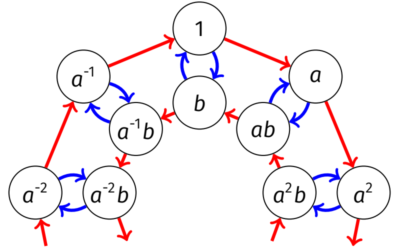

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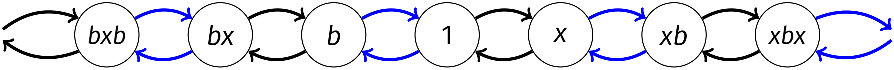

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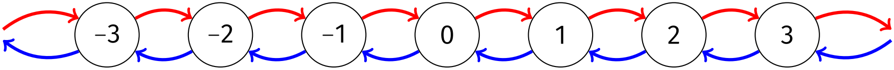
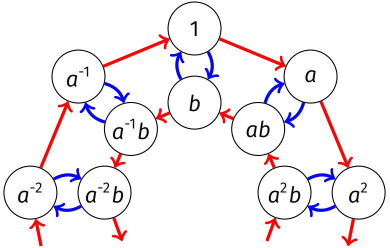


$\Gamma(\mathbb{Z}, \{\mathbf{1}, -\mathbf{1}\})$



$\Gamma(D_\infty, \{\color{red}{\boldsymbol{a}}, \color{blue}{\boldsymbol{b}}\})$

Different generating sets give different Cayley graphs

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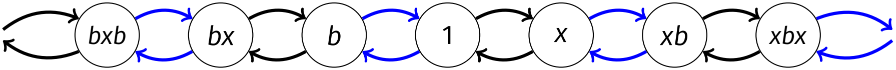
Different groups have the same Cayley graph



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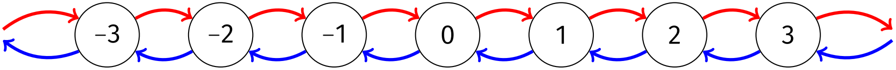

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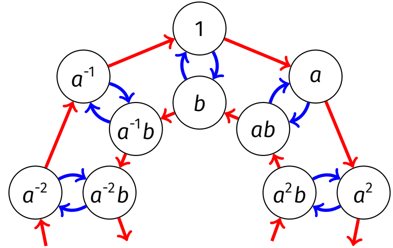

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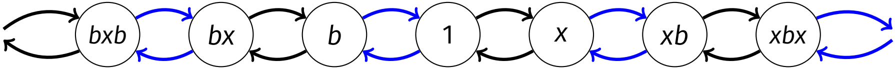
Different generating sets give different Cayley graphs

– Think up to **quasiisometry**

Different groups have the same Cayley graph

– Get used to it



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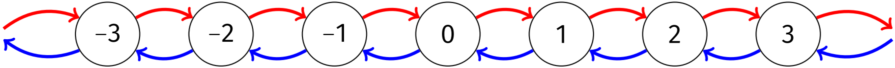

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"If H is a finite index subgroup of G, then H looks the same as G."

– a geometric group theorist

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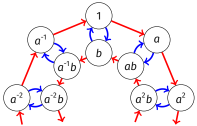

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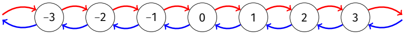

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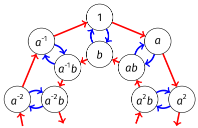

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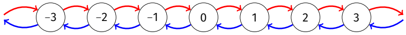

A group is **virtually** purple if it has finite index subgroup that is purple.

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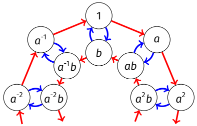

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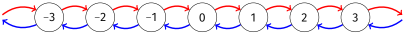

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$\langle a \rangle$ is cyclic so D_∞ is virtually cyclic

D_8

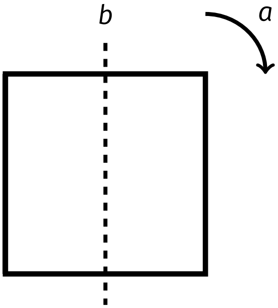
Permutation group

$$\langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle$$

Representation theory

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

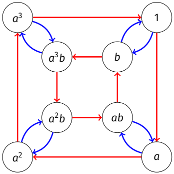
symmetries of a square



Abstract group

$\times$	1	a	a^2	a^3	b	ab	a^2b	a^3b
1	1	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	1	ab	a^2b	a^3b	b
a^2	a^2	a^3	1	a	a^2b	a^3b	b	a^2b
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	ab^3	a^2b	ab	1	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	1	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	1	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	1

Geometric group theory



Combinatorial group theory

$$D_8 = \langle a, b \mid a^4 = 1,\ b^2 = 1,\ ba = a^{-1}b \rangle$$

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a **presentation** for D_8

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$\langle a \mid a^4 = 1 \rangle$

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$\langle a \mid a^4 = 1 \rangle = C_4 \cong \mathbb{Z}/4\mathbb{Z}$

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A group is **free** if it admits a presentation of the form $\langle A \mid \ \rangle$.

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a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	ab^3	a^2b	ab	1	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	1	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	1	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	1

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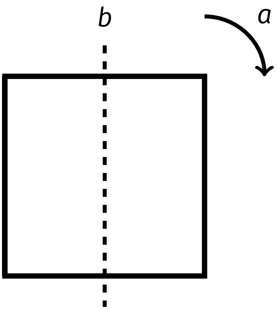

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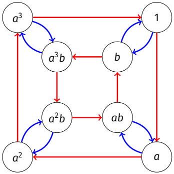

Combinatorial group theory

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The problem with presentations

Tell me something about this group

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Fix a finitely generated group $G = \langle A \rangle$. Determine whether a word in A is trivial in G.

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Theorem (Novikov, 1954)

There are groups for which the Word Problem is undecidable.

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A group G is finite $\implies$ word problem for G is solvable

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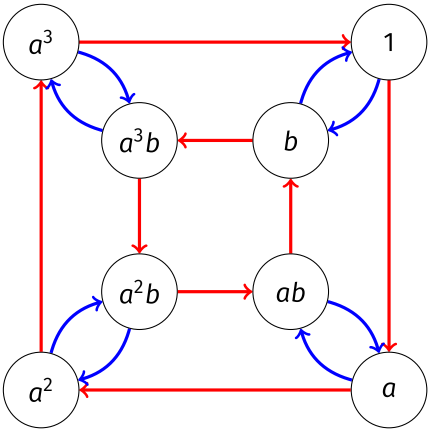

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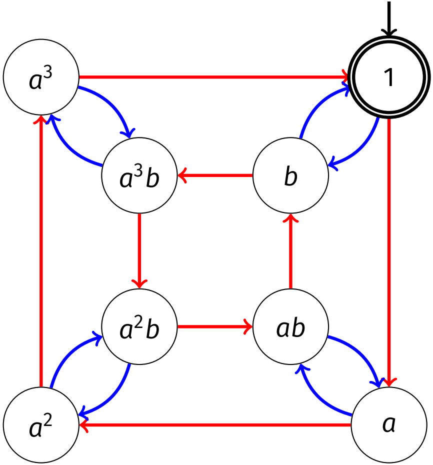

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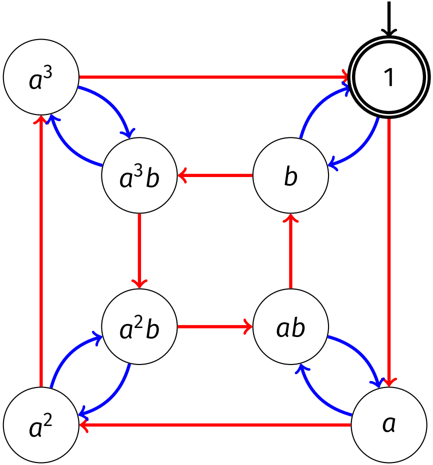

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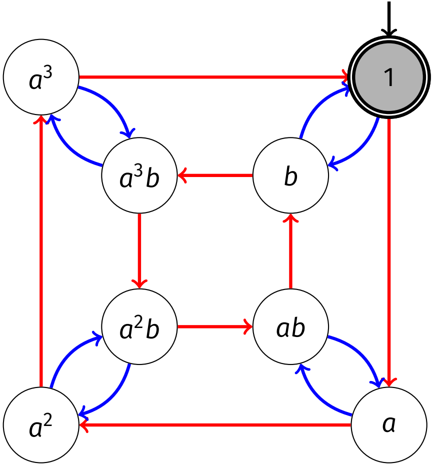

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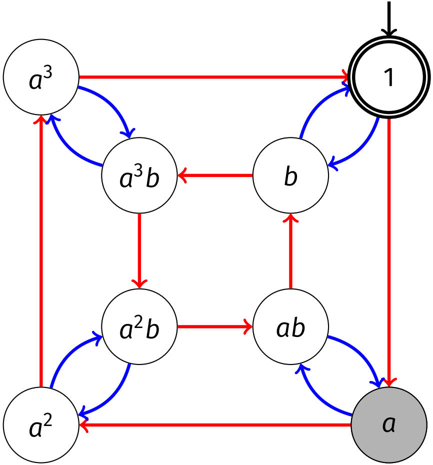

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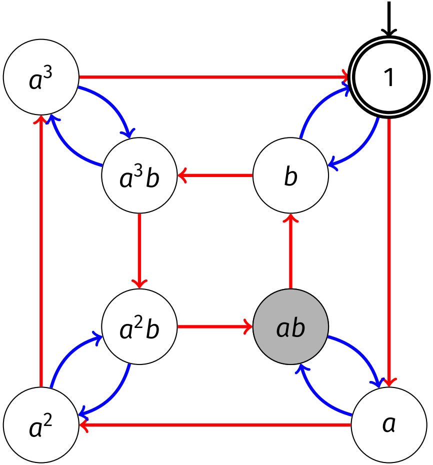

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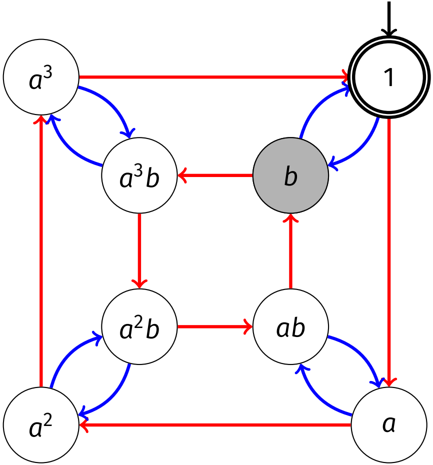

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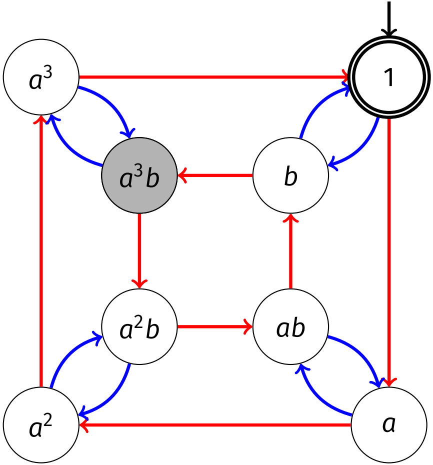

$D_8 = \langle a, b \mid a^4 = b^2 = 1,\ ba = ab^3 \rangle$ $\quad$ *abaaabbaab*

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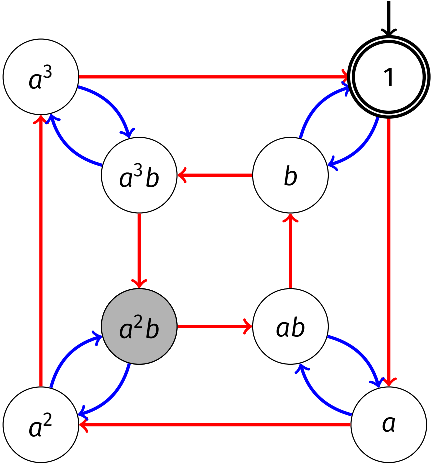

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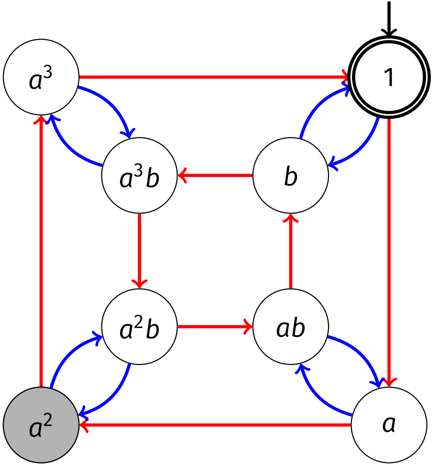

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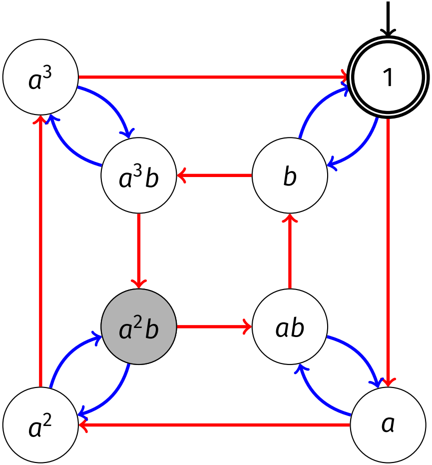

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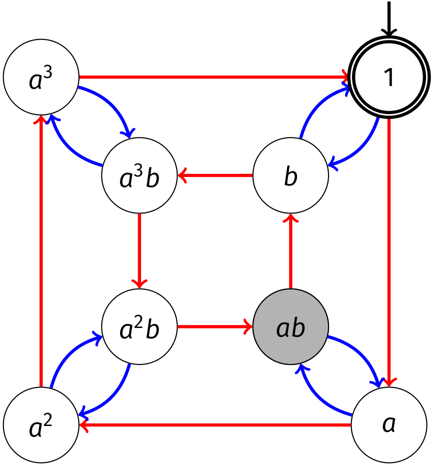

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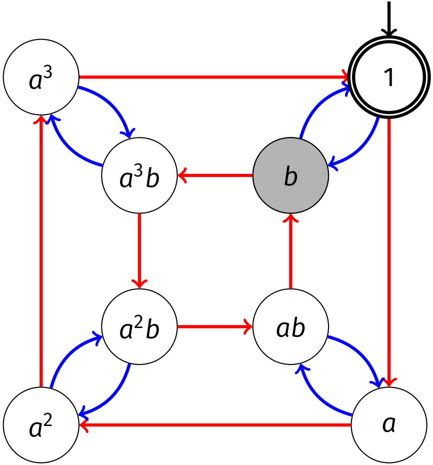

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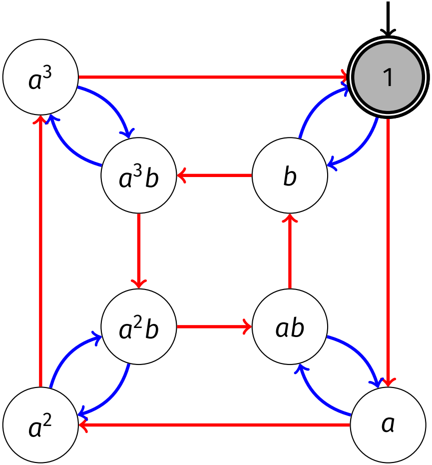

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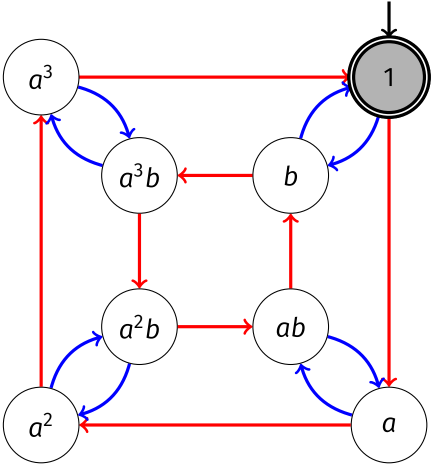

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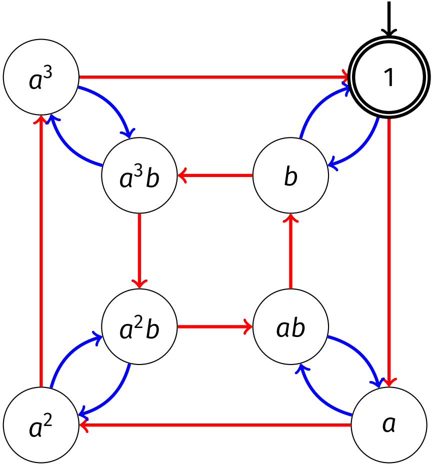
Theorem

A group G is finite $\implies$ word problem for G is solvable by an automaton



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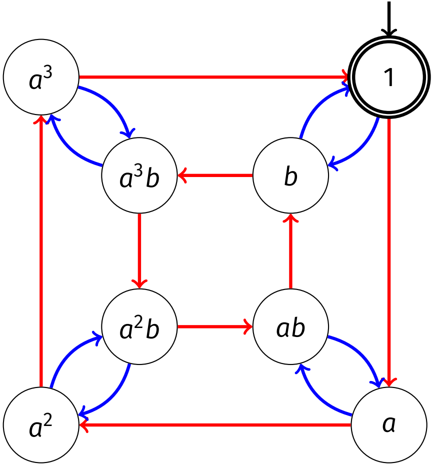

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Theorem (Anisimov, 1971)

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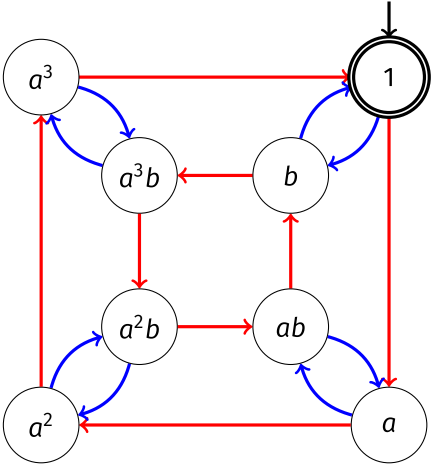

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A group G is free $\implies$ word problem for G is solvable by a stack

The category c is called a **stack** over the category C with a Grothendieck topology if it is a prestack over C and every descent datum is effective. A **descent datum** consists roughly of a covering of an object V of C by a family V_i, elements x_i in the fiber over V_i, and morphisms f_{ji} between the restrictions of x_i and x_j to $V_{ij}=V_i \times_V V_j$ satisfying the compatibility condition $f_{ki} = f_{kj}f_{ji}$. The descent datum is called **effective** if the elements x_i are essentially the pullbacks of an element x with image V.

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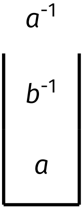

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Theorem

A group G is virtually free $\implies$ word problem for G is solvable by a push down automaton

And so our scene must to the battle fly,
Where – O for pity! – we shall much disgrace
With four or five most vile and ragged foils,
Right ill-disposed in brawl ridiculous,
The name of Agincourt. Yet sit and see,
Minding true things by what their mockeries be.

– *Henry V* (Act IV, prologue)

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Theorem (Muller–Schupp, 1983)

A group G is virtually free $\iff$ word problem for G is solvable by a push down automaton

D_8

Permutation group

$$\langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle$$

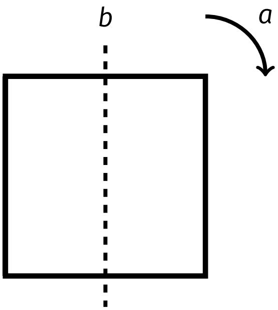
Representation theory

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

Combinatorial group theory

$$\langle a, b \mid a^4 = b^2 = 1, ba = a^{-1}b \rangle$$

symmetries of a square



Abstract group

$\times$	1	a	a^2	a^3	b	ab	a^2b	a^3b
1	1	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	1	ab	a^2b	a^3b	b
a^2	a^2	a^3	1	a	a^2b	a^3b	b	a^2b
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	ab^3	a^2b	ab	1	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	1	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	1	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	1

Geometric group theory