$\frac{3}{2}$-Generation of Groups

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featuring work with Tim Burness and with Casey Donoven

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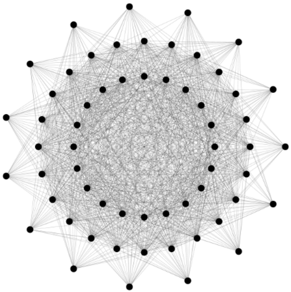

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Questions

1. What stronger properties do the finite simple groups have?
2. Which other finite groups are $\frac{3}{2}$-generated?
3. What about infinite groups?

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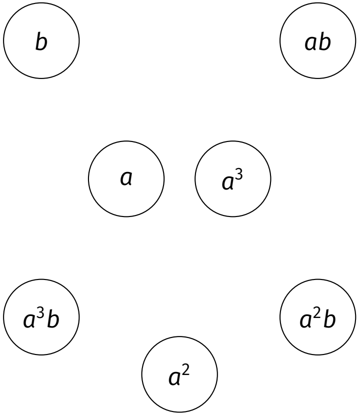

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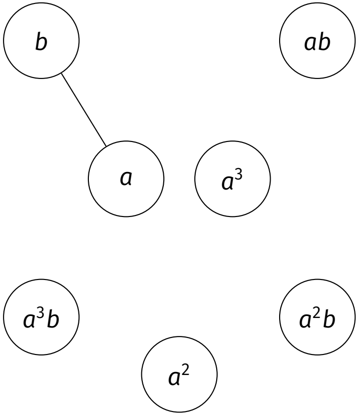

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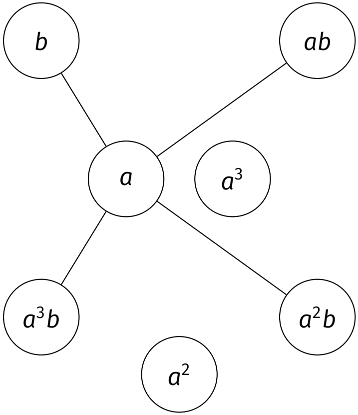

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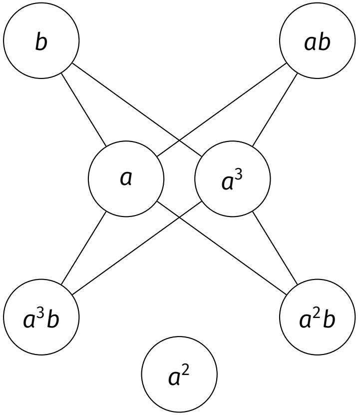

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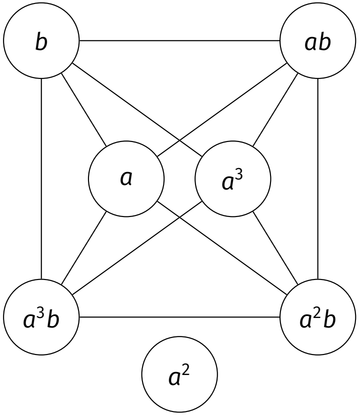

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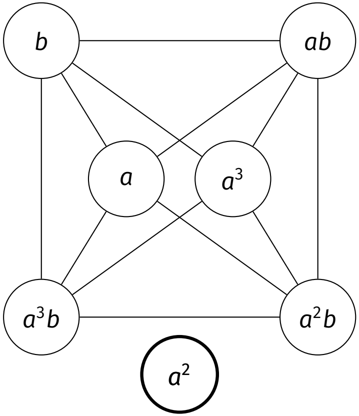

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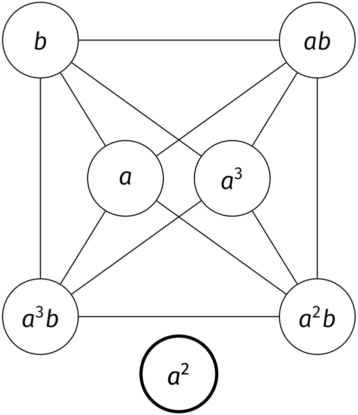

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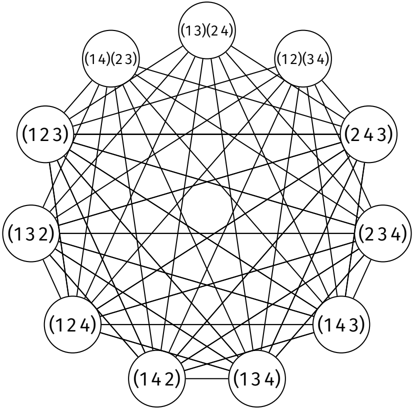

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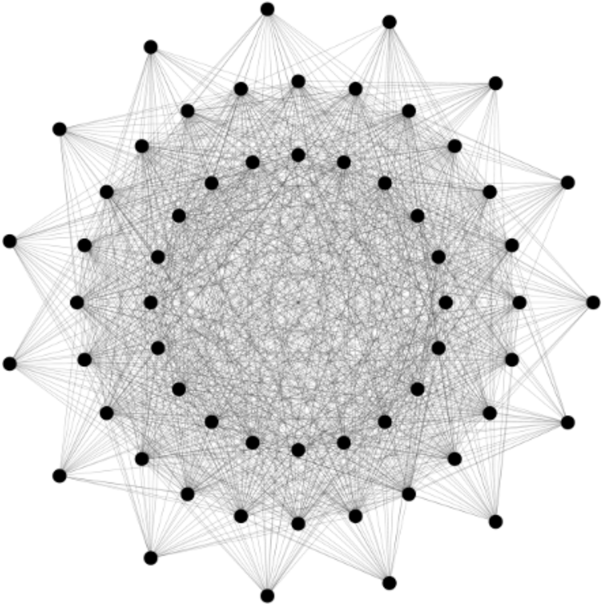

Uniform Domination of Finite Simple Groups

joint with Tim Burness

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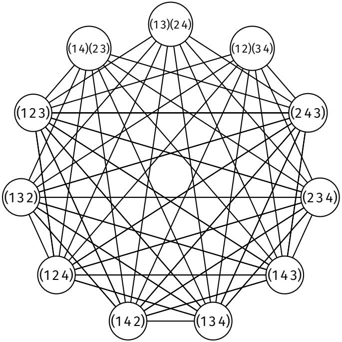

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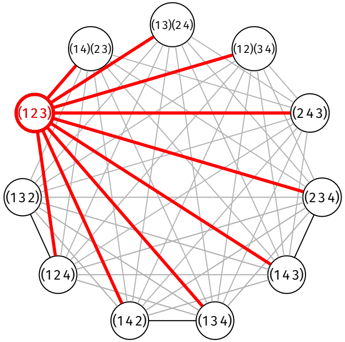

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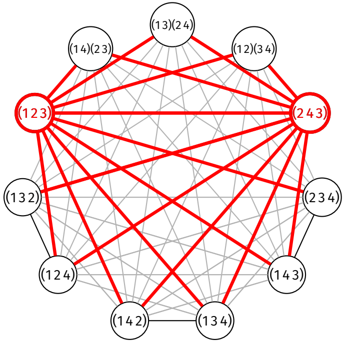

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Remark: n is even $\implies \log_2 n \leqslant \gamma_u(A_n) \leqslant 2 \log_2 n$

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$E_6, E_7, E_8, {}^2E_6$, ${}^3D_4, {}^2B_2, {}^2F_4, {}^2G_2$		F_4, G_2 ${}^2F_4(2)', {}^2G_2(3)$	5
$\mathrm{PSL}_2(q)$ ($q \geqslant 11$ odd)		$\mathrm{PSL}_2(q)$ (o/w)	3
$\mathrm{PSL}_n(q), \mathrm{PSU}_n(q)$ (n odd)		$\mathrm{PSL}_n(q), \mathrm{PSU}_n(q)$ ($n \geqslant 4$ even)	$10r + 50$
	$\mathrm{PSp}_{2r}(q)$ $\mathrm{P\Omega}^{\pm}_{2r}(q)$	$\mathrm{Sp}_{2r}(2^f), \Omega_{2r+1}(q)$	
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3. $G \in \{A_n, S_n\}$ and $\Omega = \{A \subseteq \{1, \dots, n\} \mid |A| = k\} \implies$

$$\log_2 n \leqslant b(G, \Omega) \leqslant \left\lceil \log_{\lceil n/k \rceil} n \right\rceil \cdot (\lceil n/k \rceil - 1) \quad \text{(Halasi, 2012)}.$$

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In particular, $\gamma_u(G) \leqslant b(G, G/H)$.

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Classification of Finite $\frac{3}{2}$-Generated Groups

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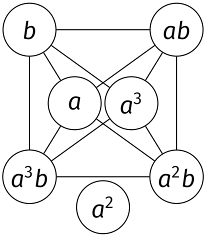
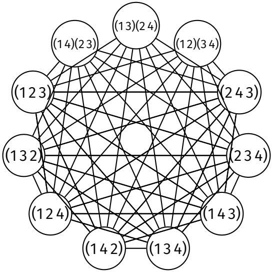
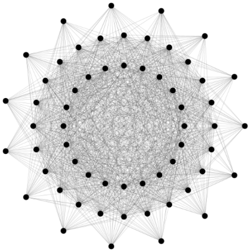


D_8



A_4



A_5

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Theorem (H, 2017 & 2019$^+$)

Let G be finite almost simple classical group. Then G is $\frac{3}{2}$-generated if and only if every proper quotient of G is cyclic.

Infinite $\frac{3}{2}$-Generated Groups

joint with Casey Donoven

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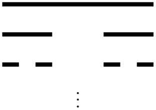

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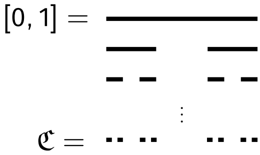

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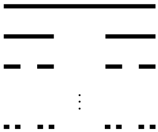

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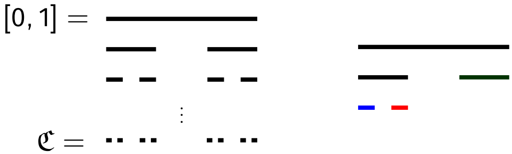

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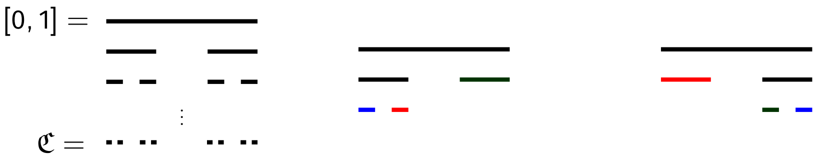

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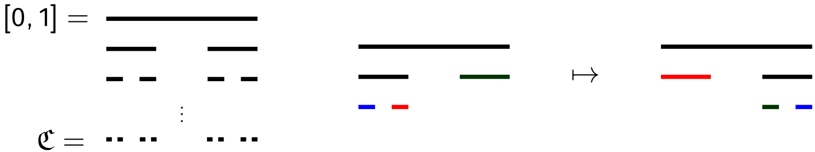

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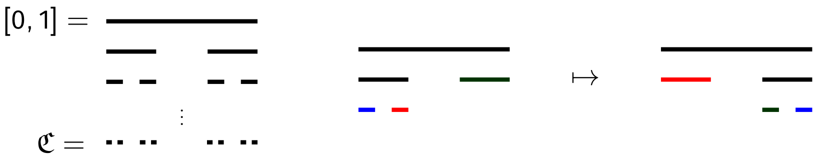

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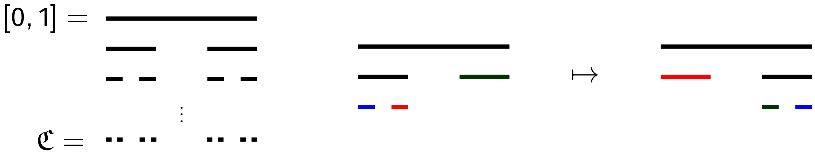

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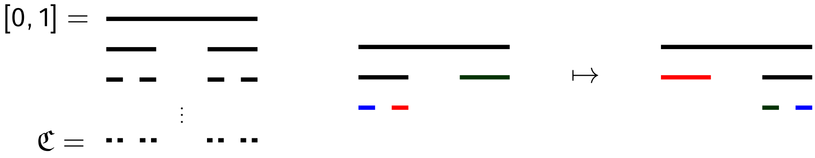

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Question

Is there a conjugacy class C of G such that for all $x \in G \setminus 1$ there exists $y \in C$ such that $G = \langle x, y \rangle$?

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