Spread, Subgroups & Shintani Descent

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featuring work with Tim Burness and Robert Guralnick

Banff International
Research Station

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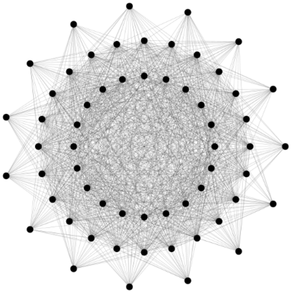

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Let G be a finite group. Every nontrivial element is contained in a generating set of minimal size if and only if $d(G/N) < d(G)$ for all $1 \neq N \trianglelefteqslant G$.

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Spread & Subgroups

"To study the **spread** of $G = \langle T, x \rangle$ with T simple and $x \in \mathrm{Aut}(T)$,
look at the maximal **subgroups** that elements of Tx are contained in."

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Lemma (Guralnick, Kantor | 2000)

We have $s(G) \geqslant k$ if there exists $s \in G$ such that for all $1 \neq x \in G$

$$\sum_{\substack{H <_{\max} G \\ s \in H}} \frac{|x^G \cap H|}{|x^G|} < \frac{1}{k}.$$

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Fix q. For $k \geqslant 1$, let T_k be a finite simple classical group with "natural module" $V_k = \mathbb{F}_q^{n_k}$ and let $x_k \in \mathrm{Aut}(T_k)$. Assume $|\langle T_k, x_k \rangle| \to \infty$. Then $s(\langle T_k, x_k \rangle)$ is bounded if and only if every element of $T_k x_k$ stabilises a 1-space of V_k.

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With previous work (e.g. [BGH | 2021] & [Guralnick, Shalev | 2003]) gives:

Theorem (H | 2021)

For $k \geqslant 1$, let T_k be a nonabelian finite simple group and let $x_k \in \mathrm{Aut}(T_k)$. Assume $|\langle T_k, x_k \rangle| \to \infty$. Then $s(\langle T_k, x_k \rangle) \not\to \infty$ if and only if $\langle T_k, x_k \rangle$ has an infinite subsequence where one of the following holds

- $\langle T_i, x_i \rangle = S_{n_i}$
- $\langle T_i, x_i \rangle = A_{n_i}$ where each n_i is divisible by a fixed prime
- T_i is $\Omega_{2m_i+1}(q)$ (q odd) or $\mathrm{Sp}_{2m_i}(q)$ (q even) for fixed q
- T_i is $\mathrm{PSL}^{\pm}_{2m_i+1}(q)$ or $\mathrm{P\Omega}^{\pm}_{2m_i}(q)$ for fixed q and x_i powering to a graph aut.

Subgroups & Shintani Descent

"To study the maximal **subgroups** that the elements of Tx are contained in, use **Shintani descent**."

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The following theorem builds on [Burness, Guest | 2013].

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Let $Y < X$ be closed connected σ_i-stable with $N_{X_{\sigma_i}}(Y_{\sigma_i}) = Y_{\sigma_i}$. Then

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Question

What about nonsingular 1-spaces? Here $Y = \mathrm{O}_{2m}$ is disconnected.

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$$\begin{array}{c} \text{the } \langle \Omega_{2m+1}(q), \varphi \rangle\text{-conjugates of} \\ N_G(\mathrm{O}^+_{2m}(q)) \text{ and } N_G(\mathrm{O}^-_{2m}(q)) \\ \text{that contain } x\varphi \end{array} \longleftrightarrow \begin{array}{c} \text{the } \Omega_{2m+1}(p)\text{-conjugates of} \\ N_{G_0}(\mathrm{O}^+_{2m}(p)) \text{ and } N_{G_0}(\mathrm{O}^-_{2m}(p)) \\ \text{that contain } F(x\varphi). \end{array}$$

Therefore, $x\varphi$ stabilises a nonsingular 1-space if and only if $F(x\varphi)$ does.

It's **not true** that $+$-type corresponds to $+$-type and $-$-type to $-$-type.

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Let $Y < X$ be closed σ_i-stable with $N_{X_{s\sigma_i}}(Y_{s\sigma_i}) = Y_{s\sigma_i}$ for $s \in N_X(Y^\circ)$. Then

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- there are infinitely many nonsimple Lie type G such that $\mu(G) = 1$.

Spread, Subgroups & Shintani Descent

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the $\langle X_{\sigma_1}, \sigma_2 \rangle$-conjugates of $\langle Y, \sigma_2 \rangle^x_{\sigma_1}$ as x ranges over X that contain g $\longleftrightarrow$ the $\langle X_{\sigma_2}, \sigma_1 \rangle$-conjugates of $\langle Y, \sigma_1 \rangle^x_{\sigma_2}$ as x ranges over X that contain $F(g)$.

Lemma (Guralnick, Kantor | 2000)

We have $s(G) \geqslant k$ if there exists $s \in G$ such that for all $1 \neq x \in G$

$$\sum_{\substack{H <_{\max} G \\ s \in H}} \frac{|x^G \cap H|}{|x^G|} < \frac{1}{k}.$$

$\downarrow$

Theorem (Burness, Guralnick, H | 2021)

We have $s(G) \geqslant 1 \iff s(G) \geqslant 2 \iff G/N$ is cyclic for all $1 \neq N \trianglelefteq G$.

$\downarrow$

Theorem (Burness, Guralnick, H | 2021)

Let G be a finite group. Every nontrivial element is contained in a generating set of minimal size if and only if $d(G/N) < d(G)$ for all $1 \neq N \trianglelefteq G$.

$\uparrow$

"only if" direction

If every nontrivial element is in a generating set of size $d(G)$, then for all $1 \leqslant N \trianglelefteq G$ and $1 \neq n \in N$, we have $G = \langle n, g_1, \dots, g_{d(G)-1} \rangle$, so $G/N = \langle Ng_1, \dots, Ng_{d(G)-1} \rangle$, so $d(G/N) < d(G)$.

$d(G) > 2$

here G is a crown-based power [Dalla Volta, Lucchini | 1985]

the theorem holds for such G [Acciarri, Lucchini | 2019]