Invariable generation and totally deranged elements of simple groups

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Let G be a finite group acting transitively with degree at least 2.
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Let G_0 be a nonabelian finite simple group. Then

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So $G_0 = A_n$ has no invariable generating set $\{x, x^a\}$ with $x \in G_0$ and $a \in G$.

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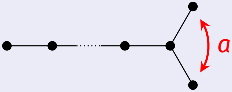

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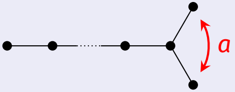

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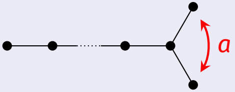

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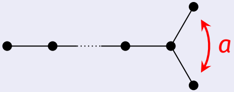

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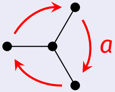

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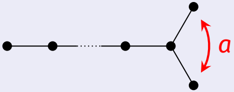

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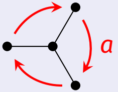

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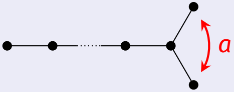

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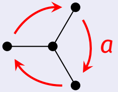

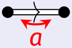

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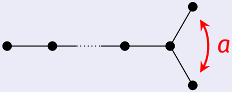

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