

Generating sets for Thompson groups

Scott Harper

Research Day

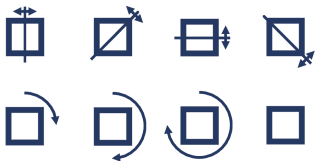
01 December 2022

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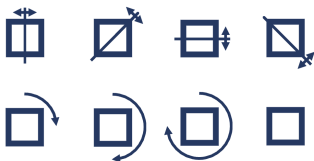


University of
St Andrews

the symmetries of a square



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The group of symmetries of a square has a **generating pair**: two elements such that any other can be obtained by repeatedly combining them.

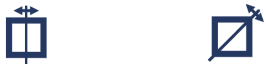
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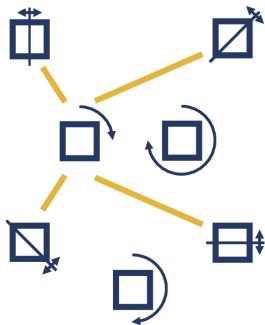
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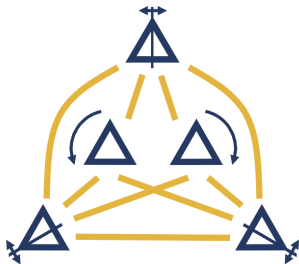
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Group of symmetries of a triangle

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Question When is every nontrivial element contained in a generating pair?

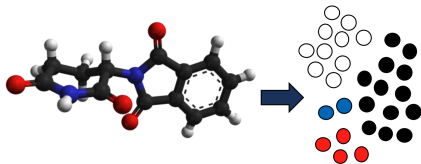
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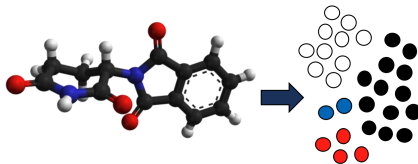
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The Periodic Table Of Finite Simple Groups

Dynkin Diagrams of Simple Lie Algebras

Class	Group	Order	Character	Group	Order	Character	Group	Order	Character	Group	Order	Character	Group	Order	Character
A_n	$A_n(2)$	$(n+1)!$	$1, \zeta, \zeta^2, \dots, \zeta^n$	$A_n(3)$	$(n+1)!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$A_n(4)$	$(n+1)!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$A_n(5)$	$(n+1)!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$A_n(7)$	$(n+1)!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$
B_n	$B_n(2)$	$2^n n!$	$1, -1, \dots, (-1)^{n-1}$	$B_n(3)$	$3^n n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$B_n(4)$	$4^n n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$B_n(5)$	$5^n n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$B_n(7)$	$7^n n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$
C_n	$C_n(2)$	$2^n n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$C_n(3)$	$3^n n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$C_n(4)$	$4^n n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$C_n(5)$	$5^n n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$C_n(7)$	$7^n n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$
D_n	$D_n(2)$	$2^{n-1} n!$	$1, -1, \dots, (-1)^{n-1}$	$D_n(3)$	$3^{n-1} n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$D_n(4)$	$4^{n-1} n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$D_n(5)$	$5^{n-1} n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$	$D_n(7)$	$7^{n-1} n!$	$1, \zeta, \zeta^2, \dots, \zeta^{n-1}$
E_6	$E_6(2)$	$2^{11} 3^6 5$	$1, \zeta, \zeta^2, \dots, \zeta^5$	$E_6(3)$	$3^{11} 5^6$	$1, \zeta, \zeta^2, \dots, \zeta^5$	$E_6(4)$	$4^{11} 5^6$	$1, \zeta, \zeta^2, \dots, \zeta^5$	$E_6(5)$	$5^{11} 7^6$	$1, \zeta, \zeta^2, \dots, \zeta^5$	$E_6(7)$	$7^{11} 11^6$	$1, \zeta, \zeta^2, \dots, \zeta^5$
E_7	$E_7(2)$	$2^{14} 3^4 5^2 7$	$1, \zeta, \zeta^2, \dots, \zeta^6$	$E_7(3)$	$3^{14} 7^4$	$1, \zeta, \zeta^2, \dots, \zeta^6$	$E_7(4)$	$4^{14} 7^4$	$1, \zeta, \zeta^2, \dots, \zeta^6$	$E_7(5)$	$5^{14} 11^4$	$1, \zeta, \zeta^2, \dots, \zeta^6$	$E_7(7)$	$7^{14} 13^4$	$1, \zeta, \zeta^2, \dots, \zeta^6$
E_8	$E_8(2)$	$2^{17} 3^3 5^3 7^2 11$	$1, \zeta, \zeta^2, \dots, \zeta^7$	$E_8(3)$	$3^{17} 11^3$	$1, \zeta, \zeta^2, \dots, \zeta^7$	$E_8(4)$	$4^{17} 11^3$	$1, \zeta, \zeta^2, \dots, \zeta^7$	$E_8(5)$	$5^{17} 13^3$	$1, \zeta, \zeta^2, \dots, \zeta^7$	$E_8(7)$	$7^{17} 17^3$	$1, \zeta, \zeta^2, \dots, \zeta^7$
F_4	$F_4(2)$	2^{26}	$1, -1, \dots, (-1)^{25}$	$F_4(3)$	3^{26}	$1, \zeta, \zeta^2, \dots, \zeta^{25}$	$F_4(4)$	4^{26}	$1, \zeta, \zeta^2, \dots, \zeta^{25}$	$F_4(5)$	5^{26}	$1, \zeta, \zeta^2, \dots, \zeta^{25}$	$F_4(7)$	7^{26}	$1, \zeta, \zeta^2, \dots, \zeta^{25}$
G_2	$G_2(2)$	$2^6 3$	$1, \zeta, \zeta^2$	$G_2(3)$	$3^6 7$	$1, \zeta, \zeta^2$	$G_2(4)$	$4^6 7$	$1, \zeta, \zeta^2$	$G_2(5)$	$5^6 7$	$1, \zeta, \zeta^2$	$G_2(7)$	$7^6 11$	$1, \zeta, \zeta^2$
H_4	$H_4(2)$	2^{12}	$1, -1, \dots, (-1)^{11}$	$H_4(3)$	3^{12}	$1, \zeta, \zeta^2, \dots, \zeta^{11}$	$H_4(4)$	4^{12}	$1, \zeta, \zeta^2, \dots, \zeta^{11}$	$H_4(5)$	5^{12}	$1, \zeta, \zeta^2, \dots, \zeta^{11}$	$H_4(7)$	7^{12}	$1, \zeta, \zeta^2, \dots, \zeta^{11}$

Legend:

- Classifying Groups
- Character Groups
- Symbol
- Order
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Theorem [BURNES, GURALNICK & H | 2021]

Let G be a finite group. Then every nontrivial element of G is contained in a generating pair iff every quotient of G other than G itself is cyclic.

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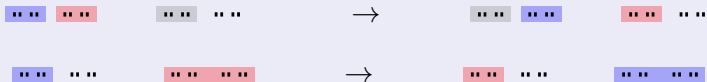
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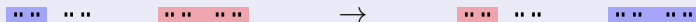
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Theorem [DONOVEN & H | 2020]

Every nontrivial element of V is contained in a generating pair.

This gave the first nontrivial example of an infinite group with this property.

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The finite simple groups have a numerous stronger generation properties.
Do these infinite groups share these properties?