Generating sets for Thompson groups

Scott Harper

Research Day

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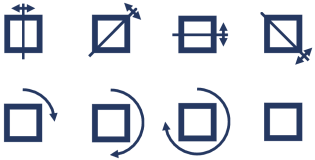
LEVERHULME TRUST

University of St Andrews

the symmetries of a square



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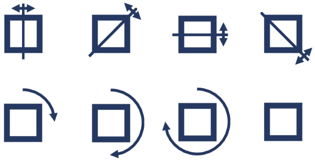

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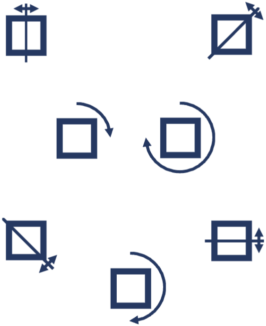

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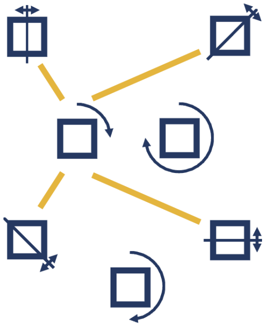

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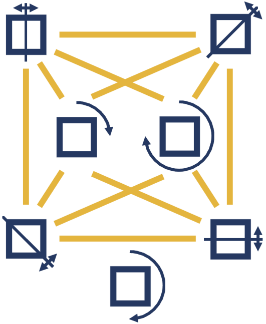

Group of symmetries of a square

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Group of symmetries of a triangle

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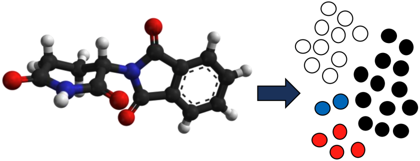

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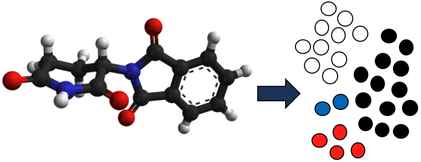
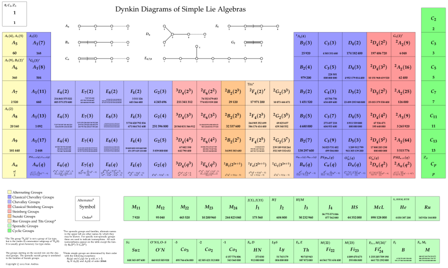


The Periodic Table Of Finite Simple Groups



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Theorem [Burness, Guralnick & H | 2021]

Let G be a finite group. Then every nontrivial element of G is contained in a generating pair iff every quotient of G other than G itself is cyclic.

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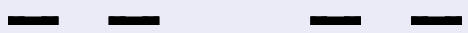

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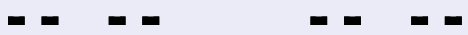

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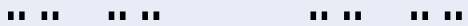
Thompson's group V is the group of prefix substitutions of Cantor space, e.g.



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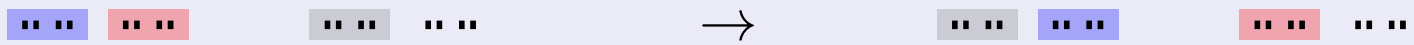

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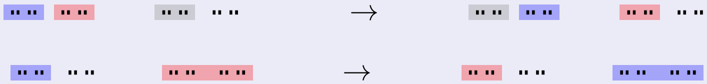

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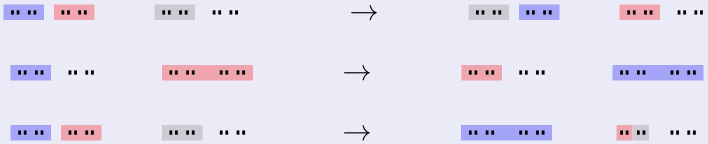

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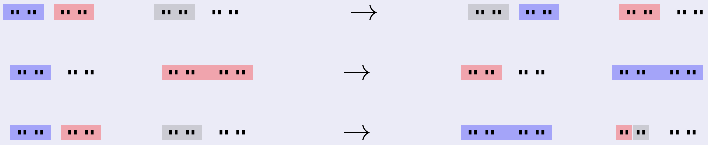

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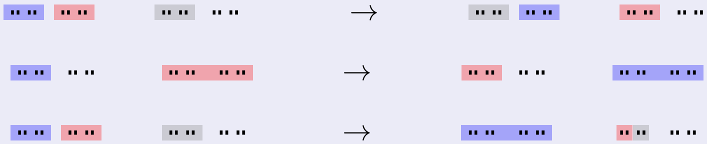

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Theorem [DONOVEN & H | 2020]

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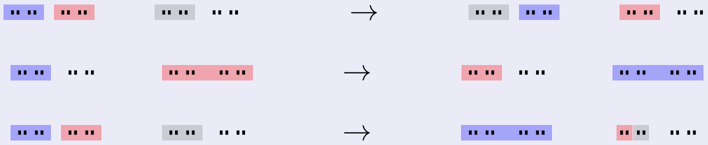

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This gave the first nontrivial example of an infinite group with this property.

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The finite simple groups have a numerous stronger generation properties.
Do these infinite groups share these properties?