How big can a minimal generating set be?

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Theorem (Whiston | 2000)

If $n \ge 3$, then $m(S_n) = n - 1$.

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Theorem (Tarski | 1975)

For all $d(G) \leqslant k \leqslant m(G)$, the group G has a minimal generating set of size k.

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If p is prime, then $2 + \omega(f) \leqslant m(\mathsf{PSL}_2(p^f)) \leqslant \max\{2 + \omega(f), 6\}$.

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If G is a finite simple group of Lie type of rank r over \mathbb{F}_{p^f} , where p is prime, then $2r + \omega(f) \leqslant m(G) \leqslant a(2r + \omega(f))^b$.

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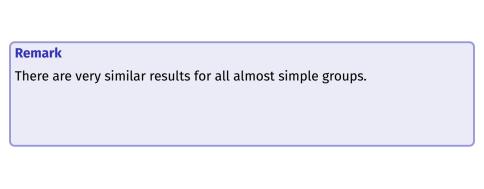
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e.g.
$$PSL_3(p^{12}) = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} y & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

where
$$\operatorname{IF}_{p^4}^\times = \langle x \rangle$$
 and $\operatorname{IF}_{p^3}^\times = \langle y \rangle$

Write $\omega(n)$ for the number of distinct prime divisors of n.



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If G is finite, then $d(G) \leqslant \max_{p \text{ prime}} d(G_p) + 1$.

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Application: a local-to-global theorem

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- Then $b(G,X) = 2 \& B(G,X) \ge n-1$. [e.g. $\{e_1, e_2, e_3, e_4\}$, $\{e_1, (e_1 + e_2), (e_1 + e_2 + e_3), (e_1 + e_2 + e_3 + e_4)\}$

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 - [e.g. $\{e_1, e_2, e_3, e_4\}$, $\{e_1, (e_1 + e_2), (e_1 + e_2 + e_3), (e_1 + e_2 + e_3 + e_4)\}$ $\{(e_1 + e_2), e_2, e_3, e_4\}$, $\{e_1, (e_2 + e_3), e_3, e_4\}$, $\{e_1, e_2, (e_3 + e_4), e_4\}$]

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Theorem (Burness et al. | 2011)

If G is a finite almost simple group with a faithful primitive nonstandard action on X, then $b(G, X) \leq 7$ with equality if and only if $G = M_{24}$ and |X| = 24.

Write I(G, X) for the maximum size of an **irredundant base** $\{x_1, \ldots, x_n\} \subseteq X$, i.e. $G > G_{(x_1)} > G_{(x_1,x_2)} > \cdots > G_{(x_1,\ldots,x_n)} = 1$. Note that $B(G,X) \leqslant I(G,X)$.

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Theorem

If G is a finite almost simple group of Lie type of rank r over \mathbb{F}_{n^f} with a faithful primitive action on X, then

(a) $I(G,X) \leqslant ar^b + \Omega(f)$ [Gill & Liebeck | 2022]

[e.g. a = 177 & b = 8]

Write I(G, X) for the maximum size of an irredundant base $\{x_1, \ldots, x_n\} \subseteq X$, i.e. $G > G_{(x_1)} > G_{(x_1,x_2)} > \cdots > G_{(x_1,\dots,x_n)} = 1$. Note that $B(G,X) \leqslant I(G,X)$.

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$$B(G,X) \leqslant ar^b + \omega(f)$$
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Notice
$$B(G, G/H)$$
 = maximum size of a subset $A \subseteq G$ such that
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Notice
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$$1 = \cap_{g \in A} H^g \text{ but } \cap_{g \in A} H^g < \cap_{g \in A'} H^g \text{ for all } A' \subsetneq A.$$

Write I(G, X) for the maximum size of an **irredundant base** $\{x_1, \ldots, x_n\} \subseteq X$, i.e. $G > G_{(x_1, x_2)} > \cdots > G_{(x_1, \ldots, x_n)} = 1$. Note that $B(G, X) \leq I(G, X)$.

Theorem

If G is a finite almost simple group of Lie type of rank r over \mathbb{F}_{p^f} with a faithful primitive action on X, then

- (a) $I(G,X) \leqslant ar^b + \Omega(f)$ [Gill & Liebeck | 2022]
- **(b)** $B(G,X) \le H(G,X) \le ar^b + \omega(f)$ [H | 2023] [e.g. a = 177 & b = 8]

Notice B(G, G/H) = maximum size of a subset $A \subseteq G$ such that $1 = \cap_{g \in A} H^g \text{ but } \cap_{g \in A} H^g < \cap_{g \in A'} H^g \text{ for all } A' \subsetneq A.$

Define H(G, G/H) = maximum size of a subset $A \subseteq G$ such that $\bigcap_{g \in A} H^g < \bigcap_{g \in A'} H^g$ for all $A' \subsetneq A$.

Let G be an almost simple group of Lie type of rank r over \mathbb{F}_{p^f} .

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For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_X < G$.

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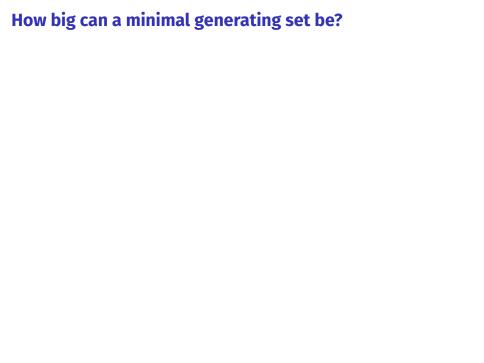
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How big can a minimal generating set be?

Theorem (H | 2023)

If G is finite, then $m(G) \leqslant a(\sum_{p \text{ prime}} d(G_p))^b$. [e.g. $a = 10^{10} \& b = 10$]

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If G is a finite almost simple group of Lie type of rank r over \mathbb{F}_{p^f} , for prime p, acting primitively on X, then $B(G,X) \leqslant ar^b + \omega(f)$. [e.g. a=177~&~b=8]