How big can a minimal generating set be?

Scott Harper

University of St Andrews

British Mathematical Colloquium

University of Manchester

17 June 2024

Generating sets

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

1 $G = C_p^n$

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

1 $G = C_p^n$ a minimal generating set is a basis of $\mathbb{F}_p^n$

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

1 $G = C_p^n$ a minimal generating set is a basis of $\mathbb{F}_p^n$

2 $G = S_n$
$(n \geqslant 3)$

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

1. $G = C_p^n$ a minimal generating set is a basis of $\mathbb{F}_p^n$
2. $G = S_n$ $(n \geqslant 3)$ $G = \langle (1\,2), (1\,2\ \ldots\ n) \rangle$

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

1. $G = C_p^n$ a minimal generating set is a basis of $\mathbb{F}_p^n$
2. $G = S_n$ $G = \langle (1\,2), (1\,2\ \dots\ n) \rangle$

   $(n \geqslant 3)$ $G = \langle (1\,2), (2\,3), (3\,4), \dots, (n\text{-}1\ \ n) \rangle$

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

Write: $\boldsymbol{d(G)}$ for the minimum size of a (minimal) generating set of G

$\boldsymbol{m(G)}$ for the maximum size of a minimal generating set of G

1 $G = C_p^n$ a minimal generating set is a basis of $\mathbb{F}_p^n$

2 $G = S_n$ $G = \langle (1\,2), (1\,2\ \ldots\ n) \rangle$

$(n \geqslant 3)$ $G = \langle (1\,2), (2\,3), (3\,4), \ldots, (n\text{-}1\ n) \rangle$

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

Write: $\boldsymbol{d(G)}$ for the minimum size of a (minimal) generating set of G

$\boldsymbol{m(G)}$ for the maximum size of a minimal generating set of G

1 $G = C_p^n$ a minimal generating set is a basis of $\mathbb{F}_p^n$ so $d(G) = m(G) = n$

2 $G = S_n$ $G = \langle (1\,2), (1\,2\,\ldots\,n) \rangle$

$(n \geqslant 3)$ $G = \langle (1\,2), (2\,3), (3\,4), \ldots, (n\text{-}1\;n) \rangle$

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

Write: $\boldsymbol{d(G)}$ for the minimum size of a (minimal) generating set of G

$\boldsymbol{m(G)}$ for the maximum size of a minimal generating set of G

1 $G = C_p^n$ a minimal generating set is a basis of $\mathbb{F}_p^n$ so $d(G) = m(G) = n$

2 $G = S_n$ $G = \langle (1\,2), (1\,2\,\ldots\,n) \rangle$ so $d(G) = 2$

$(n \geqslant 3)$ $G = \langle (1\,2), (2\,3), (3\,4), \ldots, (n\text{-}1\ n) \rangle$

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

Write: $\boldsymbol{d(G)}$ for the minimum size of a (minimal) generating set of G

$\boldsymbol{m(G)}$ for the maximum size of a minimal generating set of G

1	$G = C_p^n$	a minimal generating set is a basis of $\mathbb{F}_p^n$	so $d(G) = m(G) = n$
2	$G = S_n$	$G = \langle (1\,2), (1\,2\ \dots\ n) \rangle$	so $d(G) = 2$
	$(n \geqslant 3)$	$G = \langle (1\,2), (2\,3), (3\,4), \dots, (n\text{-}1\ n) \rangle$	so $m(G) \geqslant n - 1$

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

Write: $\boldsymbol{d(G)}$ for the minimum size of a (minimal) generating set of G

$\boldsymbol{m(G)}$ for the maximum size of a minimal generating set of G

1 $G = C_p^n$ a minimal generating set is a basis of $\mathbb{F}_p^n$ so $d(G) = m(G) = n$

2 $G = S_n$ $G = \langle (1\,2), (1\,2\,\ldots\,n) \rangle$ so $d(G) = 2$

$(n \geqslant 3)$ $G = \langle (1\,2), (2\,3), (3\,4), \ldots, (n\text{-}1\;\, n) \rangle$ so $m(G) \geqslant n - 1$

Theorem (Whiston | 2000)

If $n \geqslant 3$, then $m(S_n) = n - 1$.

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

Write: $\boldsymbol{d(G)}$ for the minimum size of a (minimal) generating set of G

$\boldsymbol{m(G)}$ for the maximum size of a minimal generating set of G

1	$G = C_p^n$	a minimal generating set is a basis of $\mathbb{F}_p^n$	so $d(G) = m(G) = n$
2	$G = S_n$	$G = \langle (1\,2), (1\,2\ \ldots\ n) \rangle$	so $d(G) = 2$
	$(n \geqslant 3)$	$G = \langle (1\,2), (2\,3), (3\,4), \ldots, (n\text{-}1\ n) \rangle$	so $m(G) \geqslant n - 1$

Theorem (Whiston | 2000)

If $n \geqslant 3$, then $m(S_n) = n - 1$ and $m(A_n) = n - 2$.

Generating sets

Let G be a finite group.

A generating set of G is a **minimal** if no proper subset generates G.

Write: $\boldsymbol{d(G)}$ for the minimum size of a (minimal) generating set of G

$\boldsymbol{m(G)}$ for the maximum size of a minimal generating set of G

1 $G = C_p^n$ a minimal generating set is a basis of $\mathbb{F}_p^n$ so $d(G) = m(G) = n$

2 $G = S_n$ $G = \langle (1\,2), (1\,2\ \dots\ n) \rangle$ so $d(G) = 2$

$(n \geqslant 3)$ $G = \langle (1\,2), (2\,3), (3\,4), \dots, (n\text{-}1\ n) \rangle$ so $m(G) \geqslant n - 1$

Theorem (Whiston | 2000)

If $n \geqslant 3$, then $m(S_n) = n - 1$ and $m(A_n) = n - 2$.

Theorem (Tarski | 1975)

For all $d(G) \leqslant k \leqslant m(G)$, the group G has a minimal generating set of size k.

Theorem (Steinberg | 1962 + CFSG)

If G is a finite simple group, then $d(G) \leqslant 2$.

Theorem (Steinberg | 1962 + CFSG)

If G is a finite simple group, then $d(G) \leqslant 2$.

Theorem (Whiston & Saxl | 2002)

If p is prime, then $2 + \omega(f) \leqslant m(\mathrm{PSL}_2(p^f)) \leqslant \max\{2 + \omega(f), 6\}$.

Theorem (Steinberg | 1962 + CFSG)

If G is a finite simple group, then $d(G) \leqslant 2$.

Theorem (Whiston & Saxl | 2002)

If p is prime, then $2 + \omega(f) \leqslant m(\mathrm{PSL}_2(p^f)) \leqslant \max\{2 + \omega(f), 6\}$.

Write $\omega(n)$ for the number of distinct prime divisors of n.

Theorem (Steinberg | 1962 + CFSG)

If G is a finite simple group, then $d(G) \leqslant 2$.

Theorem (Whiston & Saxl | 2002)

If p is prime, then $2 + \omega(f) \leqslant m(\mathrm{PSL}_2(p^f)) \leqslant \max\{2 + \omega(f), 6\}$.

Theorem (H | 2023)

If G is a finite simple group of Lie type of rank r over $\mathbb{F}_{p^f}$, where p is prime, then $2r + \omega(f) \leqslant m(G) \leqslant a(2r + \omega(f))^b$.

Write $\omega(n)$ for the number of distinct prime divisors of n.

Theorem (Steinberg | 1962 + CFSG)

If G is a finite simple group, then $d(G) \leqslant 2$.

Theorem (Whiston & Saxl | 2002)

If p is prime, then $2 + \omega(f) \leqslant m(\mathrm{PSL}_2(p^f)) \leqslant \max\{2 + \omega(f), 6\}$.

Theorem (H | 2023)

If G is a finite simple group of Lie type of rank r over $\mathbb{F}_{p^f}$, where p is prime, then $2r + \omega(f) \leqslant m(G) \leqslant a(2r + \omega(f))^b$. [e.g. $a = 10^5$ & $b = 10$]

Write $\omega(n)$ for the number of distinct prime divisors of n.

Theorem (Steinberg | 1962 + CFSG)

If G is a finite simple group, then $d(G) \leqslant 2$.

Theorem (Whiston & Saxl | 2002)

If p is prime, then $2 + \omega(f) \leqslant m(\mathrm{PSL}_2(p^f)) \leqslant \max\{2 + \omega(f), 6\}$.

Theorem (H | 2023)

If G is a finite simple group of Lie type of rank r over $\mathbb{F}_{p^f}$, where p is prime, then $2r + \omega(f) \leqslant m(G) \leqslant a(2r + \omega(f))^b$. [e.g. $a = 10^5$ & $b = 10$]

e.g. $$\mathrm{PSL}_3(p^{12}) = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} y & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

where $\mathbb{F}_{p^4}^{\times} = \langle x \rangle$ and $\mathbb{F}_{p^3}^{\times} = \langle y \rangle$

Write $\omega(n)$ for the number of distinct prime divisors of n.

Remark

There are very similar results for all almost simple groups.

Remark

There are very similar results for all almost simple groups.

A group G is **almost simple** if $G_0 \leqslant G \leqslant \mathrm{Aut}(G_0)$ for nonabelian simple G_0.

Remark

There are very similar results for all almost simple groups.

A group G is **almost simple** if $G_0 \leqslant G \leqslant \mathrm{Aut}(G_0)$ for nonabelian simple G_0.

e.g. $G_0 = A_n$ and $G = S_n$,

Remark

There are very similar results for all almost simple groups.

A group G is **almost simple** if $G_0 \leqslant G \leqslant \mathrm{Aut}(G_0)$ for nonabelian simple G_0.

e.g. $G_0 = A_n$ and $G = S_n$, $G_0 = \mathrm{PSL}_n(q)$ and $G = \mathrm{PGL}_n(q)$,

Remark

There are very similar results for all almost simple groups.

A group G is **almost simple** if $G_0 \leqslant G \leqslant \mathrm{Aut}(G_0)$ for nonabelian simple G_0.

e.g. $G_0 = A_n$ and $G = S_n$, $G_0 = \mathrm{PSL}_n(q)$ and $G = \mathrm{PGL}_n(q)$, $G_0 = G =$ 

Application: a local-to-global theorem

Application: a local-to-global theorem

Question Are $d(G)$ and $m(G)$ controlled by the Sylow subgroups of G?

Application: a local-to-global theorem

Question Are $d(G)$ and $m(G)$ controlled by the Sylow subgroups of G?

G is C_p^n $\quad$ $d(G) = m(G) = n$

Application: a local-to-global theorem

Question Are $d(G)$ and $m(G)$ controlled by the Sylow subgroups of G?

G is C_p^n	$d(G) = m(G) = n$
G is a p-group	$d(G) = m(G)$

Application: a local-to-global theorem

Question Are $d(G)$ and $m(G)$ controlled by the Sylow subgroups of G?

G is C_p^n	$d(G) = m(G) = n$
G is a p-group	$d(G) = m(G)$
G is nilpotent	

Application: a local-to-global theorem

Question Are $d(G)$ and $m(G)$ controlled by the Sylow subgroups of G?

G is C_p^n	$d(G) = m(G) = n$
G is a p-group	$d(G) = m(G)$
G is nilpotent	

[$G = G_{p_1} \times \cdots \times G_{p_k}$ where G_p is a Sylow p-subgroup]

Application: a local-to-global theorem

Question Are $d(G)$ and $m(G)$ controlled by the Sylow subgroups of G?

G is C_p^n	$d(G) = m(G) = n$
G is a p-group	$d(G) = m(G)$
G is nilpotent	$d(G) = \max_{p \text{ prime}} d(G_p)$ & $m(G) = \sum_{p \text{ prime}} d(G_p)$
	[$G = G_{p_1} \times \cdots \times G_{p_k}$ where G_p is a Sylow p-subgroup]

Application: a local-to-global theorem

Question Are $d(G)$ and $m(G)$ controlled by the Sylow subgroups of G?

G is C_p^n	$d(G) = m(G) = n$
G is a p-group	$d(G) = m(G)$
G is nilpotent	$d(G) = \max_{p \text{ prime}} d(G_p)$ & $m(G) = \sum_{p \text{ prime}} d(G_p)$
	[$G = G_{p_1} \times \cdots \times G_{p_k}$ where G_p is a Sylow p-subgroup]

Theorem (Guralnick // Lucchini | 1989)

If G is finite, then $d(G) \leqslant \max_{p \text{ prime}} d(G_p) + 1$.

Application: a local-to-global theorem

Question Are $d(G)$ and $m(G)$ controlled by the Sylow subgroups of G?

G is C_p^n	$d(G) = m(G) = n$
G is a p-group	$d(G) = m(G)$
G is nilpotent	$d(G) = \max_{p \text{ prime}} d(G_p)$ & $m(G) = \sum_{p \text{ prime}} d(G_p)$
	[$G = G_{p_1} \times \cdots \times G_{p_k}$ where G_p is a Sylow p-subgroup]

Theorem (Guralnick // Lucchini | 1989)

If G is finite, then $d(G) \leqslant \max_{p \text{ prime}} d(G_p) + 1$.

However, the bound $m(G) \leqslant \sum_{p \text{ prime}} d(G_p) + 1$ is true for soluble groups G but is false in general [Lucchini, Moscatiello, Spiga | 2021].

Application: a local-to-global theorem

Question Are $d(G)$ and $m(G)$ controlled by the Sylow subgroups of G?

G is C_p^n $d(G) = m(G) = n$

G is a p-group $d(G) = m(G)$

G is nilpotent $d(G) = \max_{p \text{ prime}} d(G_p)$ & $m(G) = \sum_{p \text{ prime}} d(G_p)$

[$G = G_{p_1} \times \cdots \times G_{p_k}$ where G_p is a Sylow p-subgroup]

Theorem (Guralnick // Lucchini | 1989)

If G is finite, then $d(G) \leqslant \max_{p \text{ prime}} d(G_p) + 1$.

However, the bound $m(G) \leqslant \sum_{p \text{ prime}} d(G_p) + 1$ is true for soluble groups G but is false in general [Lucchini, Moscatiello, Spiga | 2021].

Theorem (H | 2023)

If G is finite, then $m(G) \leqslant a(\sum_{p \text{ prime}} d(G_p))^b$. [e.g. $a = 10^{10}$ & $b = 10$]

Bases for permutation groups

Bases for permutation groups

Let G act faithfully on a finite set X.

Bases for permutation groups

Let G act faithfully on a finite set X.

A subset $\{x_1, \ldots, x_n\} \subseteq X$ is a **base** if the pointwise stabiliser $G_{(x_1,\ldots,x_n)}$ is 1.

Bases for permutation groups

Let G act faithfully on a finite set X.

A subset $\{x_1, \ldots, x_n\} \subseteq X$ is a **base** if the pointwise stabiliser $G_{(x_1,\ldots,x_n)}$ is 1.

Write: $\boldsymbol{b(G, X)}$ for the minimum size of a (minimal) base for G on X

$\boldsymbol{B(G, X)}$ for the maximum size of a minimal base for G on X

Bases for permutation groups

Let G act faithfully on a finite set X.

A subset $\{x_1, \ldots, x_n\} \subseteq X$ is a **base** if the pointwise stabiliser $G_{(x_1,\ldots,x_n)}$ is 1.

Write: $\boldsymbol{b(G, X)}$ for the minimum size of a (minimal) base for G on X

$\boldsymbol{B(G, X)}$ for the maximum size of a minimal base for G on X

1 Let $G = \mathrm{GL}_n(F)$ act on $X = F^n$. Then a minimal base is just a basis of F^n.

Bases for permutation groups

Let G act faithfully on a finite set X.

A subset $\{x_1, \ldots, x_n\} \subseteq X$ is a **base** if the pointwise stabiliser $G_{(x_1,\ldots,x_n)}$ is 1.

Write: $\boldsymbol{b(G, X)}$ for the minimum size of a (minimal) base for G on X

$\boldsymbol{B(G, X)}$ for the maximum size of a minimal base for G on X

1 Let $G = \mathrm{GL}_n(F)$ act on $X = F^n$. Then a minimal base is just a basis of F^n. Therefore $b(G, X) = B(G, X) = n$.

Bases for permutation groups

Let G act faithfully on a finite set X.

A subset $\{x_1, \ldots, x_n\} \subseteq X$ is a **base** if the pointwise stabiliser $G_{(x_1,\ldots,x_n)}$ is 1.

Write: $\boldsymbol{b(G, X)}$ for the minimum size of a (minimal) base for G on X

$\boldsymbol{B(G, X)}$ for the maximum size of a minimal base for G on X

1. Let $G = \mathrm{GL}_n(F)$ act on $X = F^n$. Then a minimal base is just a basis of F^n. Therefore $b(G, X) = B(G, X) = n$.
2. Let $G = \mathrm{PGL}_n(F)$ act on $X = \{\{\langle v_1\rangle, \ldots, \langle v_n\rangle\} \mid \{v_1, \ldots, v_n\}$ a basis of $F^n\}$.

Bases for permutation groups

Let G act faithfully on a finite set X.

A subset $\{x_1, \ldots, x_n\} \subseteq X$ is a **base** if the pointwise stabiliser $G_{(x_1,\ldots,x_n)}$ is 1.

Write: $\boldsymbol{b(G,X)}$ for the minimum size of a (minimal) base for G on X

$\boldsymbol{B(G,X)}$ for the maximum size of a minimal base for G on X

1. Let $G = \mathrm{GL}_n(F)$ act on $X = F^n$. Then a minimal base is just a basis of F^n. Therefore $b(G,X) = B(G,X) = n$.
2. Let $G = \mathrm{PGL}_n(F)$ act on $X = \{\{\langle v_1\rangle, \ldots, \langle v_n\rangle\} \mid \{v_1, \ldots, v_n\}$ a basis of $F^n\}$. Then $b(G,X) = 2$ & $B(G,X) \geqslant n-1$.

Bases for permutation groups

Let G act faithfully on a finite set X.

A subset $\{x_1, \ldots, x_n\} \subseteq X$ is a **base** if the pointwise stabiliser $G_{(x_1,\ldots,x_n)}$ is 1.

Write: $\boldsymbol{b(G,X)}$ for the minimum size of a (minimal) base for G on X

$\boldsymbol{B(G,X)}$ for the maximum size of a minimal base for G on X

1 Let $G = \mathrm{GL}_n(F)$ act on $X = F^n$. Then a minimal base is just a basis of F^n.
Therefore $b(G,X) = B(G,X) = n$.

2 Let $G = \mathrm{PGL}_n(F)$ act on $X = \{\{\langle v_1\rangle, \ldots, \langle v_n\rangle\} \mid \{v_1, \ldots, v_n\}$ a basis of $F^n\}$.
Then $b(G,X) = 2$ & $B(G,X) \geqslant n-1$.
[**e.g.** $\{e_1, e_2, e_3, e_4\}$, $\{e_1, (e_1+e_2), (e_1+e_2+e_3), (e_1+e_2+e_3+e_4)\}$

Bases for permutation groups

Let G act faithfully on a finite set X.

A subset $\{x_1, \ldots, x_n\} \subseteq X$ is a **base** if the pointwise stabiliser $G_{(x_1,\ldots,x_n)}$ is 1.

Write: $\boldsymbol{b(G,X)}$ for the minimum size of a (minimal) base for G on X

$\boldsymbol{B(G,X)}$ for the maximum size of a minimal base for G on X

1 Let $G = \mathrm{GL}_n(F)$ act on $X = F^n$. Then a minimal base is just a basis of F^n.
Therefore $b(G,X) = B(G,X) = n$.

2 Let $G = \mathrm{PGL}_n(F)$ act on $X = \{\{\langle v_1\rangle, \ldots, \langle v_n\rangle\} \mid \{v_1, \ldots, v_n\}$ a basis of $F^n\}$.
Then $b(G,X) = 2$ & $B(G,X) \geqslant n-1$.
[**e.g.** $\{e_1, e_2, e_3, e_4\}$, $\{e_1, (e_1+e_2), (e_1+e_2+e_3), (e_1+e_2+e_3+e_4)\}$
$\{(e_1+e_2), e_2, e_3, e_4\}$, $\{e_1, (e_2+e_3), e_3, e_4\}$, $\{e_1, e_2, (e_3+e_4), e_4\}$]

Bases for permutation groups

Let G act faithfully on a finite set X.

A subset $\{x_1, \ldots, x_n\} \subseteq X$ is a **base** if the pointwise stabiliser $G_{(x_1,\ldots,x_n)}$ is 1.

Write: $\boldsymbol{b(G,X)}$ for the minimum size of a (minimal) base for G on X

$\boldsymbol{B(G,X)}$ for the maximum size of a minimal base for G on X

1. Let $G = \mathrm{GL}_n(F)$ act on $X = F^n$. Then a minimal base is just a basis of F^n.
   Therefore $b(G,X) = B(G,X) = n$.
2. Let $G = \mathrm{PGL}_n(F)$ act on $X = \{\{\langle v_1\rangle, \ldots, \langle v_n\rangle\} \mid \{v_1, \ldots, v_n\}$ a basis of $F^n\}$.
   Then $b(G,X) = 2$ & $B(G,X) \geqslant n-1$.
   [**e.g.** $\{e_1, e_2, e_3, e_4\}$, $\{e_1, (e_1+e_2), (e_1+e_2+e_3), (e_1+e_2+e_3+e_4)\}$
   $\{(e_1+e_2), e_2, e_3, e_4\}$, $\{e_1, (e_2+e_3), e_3, e_4\}$, $\{e_1, e_2, (e_3+e_4), e_4\}$]

Theorem (Burness et al. | 2011)

If G is a finite almost simple group with a faithful primitive nonstandard action on X, then $b(G,X) \leqslant 7$ with equality if and only if $G = \mathrm{M}_{24}$ and $|X| = 24$.

Write: $\omega(n)$ for the number of distinct prime divisors of n

$\Omega(n)$ for the number of prime divisors of n with multiplicity

Write: $\omega(n)$ for the number of distinct prime divisors of n

$\Omega(n)$ for the number of prime divisors of n with multiplicity

Write $I(G, X)$ for the maximum size of an **irredundant base** $\{x_1, \ldots, x_n\} \subseteq X$, i.e. $G > G_{(x_1)} > G_{(x_1,x_2)} > \cdots > G_{(x_1,\ldots,x_n)} = 1$. Note that $B(G, X) \leqslant I(G, X)$.

Write: $\omega(n)$ for the number of distinct prime divisors of n

$\Omega(n)$ for the number of prime divisors of n with multiplicity

Write $I(G, X)$ for the maximum size of an **irredundant base** $\{x_1, \ldots, x_n\} \subseteq X$, i.e. $G > G_{(x_1)} > G_{(x_1,x_2)} > \cdots > G_{(x_1,\ldots,x_n)} = 1$. Note that $B(G, X) \leqslant I(G, X)$.

Theorem

If G is a finite almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$ with a faithful primitive action on X, then

(a) $I(G, X) \leqslant ar^b + \Omega(f)$ [Gill & Liebeck | 2022]

[e.g. $a = 177$ & $b = 8$]

Write: $\omega(n)$ for the number of distinct prime divisors of n

$\Omega(n)$ for the number of prime divisors of n with multiplicity

Write $I(G, X)$ for the maximum size of an **irredundant base** $\{x_1, \dots, x_n\} \subseteq X$, i.e. $G > G_{(x_1)} > G_{(x_1,x_2)} > \cdots > G_{(x_1,\dots,x_n)} = 1$. Note that $B(G, X) \leqslant I(G, X)$.

Theorem

If G is a finite almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$ with a faithful primitive action on X, then

(a) $I(G, X) \leqslant ar^b + \Omega(f)$ [Gill & Liebeck | 2022]

(b) $B(G, X) \leqslant ar^b + \omega(f)$ [H | 2023] [e.g. $a = 177$ & $b = 8$]

Write: $\omega(n)$ for the number of distinct prime divisors of n

$\Omega(n)$ for the number of prime divisors of n with multiplicity

Write $I(G, X)$ for the maximum size of an **irredundant base** $\{x_1, \ldots, x_n\} \subseteq X$, i.e. $G > G_{(x_1)} > G_{(x_1,x_2)} > \cdots > G_{(x_1,\ldots,x_n)} = 1$. Note that $B(G, X) \leqslant I(G, X)$.

Theorem

If G is a finite almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$ with a faithful primitive action on X, then

(a) $I(G, X) \leqslant ar^b + \Omega(f)$ [Gill & Liebeck | 2022]

(b) $B(G, X) \leqslant H(G, X) \leqslant ar^b + \omega(f)$ [H | 2023] [e.g. $a = 177$ & $b = 8$]

Write: $\omega(n)$ for the number of distinct prime divisors of n
$\Omega(n)$ for the number of prime divisors of n with multiplicity

Write $I(G, X)$ for the maximum size of an **irredundant base** $\{x_1, \ldots, x_n\} \subseteq X$, i.e. $G > G_{(x_1)} > G_{(x_1,x_2)} > \cdots > G_{(x_1,\ldots,x_n)} = 1$. Note that $B(G,X) \leqslant I(G,X)$.

Theorem

If G is a finite almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$ with a faithful primitive action on X, then

(a) $I(G,X) \leqslant ar^b + \Omega(f)$ [Gill & Liebeck | 2022]

(b) $B(G,X) \leqslant H(G,X) \leqslant ar^b + \omega(f)$ [H | 2023] [e.g. $a = 177$ & $b = 8$]

Notice $B(G, G/H)$ = maximum size of a subset $A \subseteq G$ such that

$$1 = \cap_{g \in A} H^g$$

Write: $\omega(n)$ for the number of distinct prime divisors of n

$\Omega(n)$ for the number of prime divisors of n with multiplicity

Write $I(G, X)$ for the maximum size of an **irredundant base** $\{x_1, \ldots, x_n\} \subseteq X$, i.e. $G > G_{(x_1)} > G_{(x_1,x_2)} > \cdots > G_{(x_1,\ldots,x_n)} = 1$. Note that $B(G, X) \leqslant I(G, X)$.

Theorem

If G is a finite almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$ with a faithful primitive action on X, then

(a) $I(G, X) \leqslant ar^b + \Omega(f)$ [Gill & Liebeck | 2022]

(b) $B(G, X) \leqslant H(G, X) \leqslant ar^b + \omega(f)$ [H | 2023] [e.g. $a = 177$ & $b = 8$]

Notice $B(G, G/H)$ = maximum size of a subset $A \subseteq G$ such that

$$1 = \cap_{g \in A} H^g \quad \text{but} \quad \cap_{g \in A} H^g < \cap_{g \in A'} H^g \quad \text{for all } A' \subsetneq A.$$

Write: $\omega(n)$ for the number of distinct prime divisors of n
$\Omega(n)$ for the number of prime divisors of n with multiplicity

Write $I(G, X)$ for the maximum size of an **irredundant base** $\{x_1, \ldots, x_n\} \subseteq X$, i.e. $G > G_{(x_1)} > G_{(x_1,x_2)} > \cdots > G_{(x_1,\ldots,x_n)} = 1$. Note that $B(G, X) \leqslant I(G, X)$.

Theorem

If G is a finite almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$ with a faithful primitive action on X, then

(a) $I(G, X) \leqslant ar^b + \Omega(f)$ [Gill & Liebeck | 2022]

(b) $B(G, X) \leqslant H(G, X) \leqslant ar^b + \omega(f)$ [H | 2023] [e.g. $a = 177$ & $b = 8$]

Notice $B(G, G/H)$ = maximum size of a subset $A \subseteq G$ such that

$$1 = \cap_{g \in A} H^g \text{ but } \cap_{g \in A} H^g < \cap_{g \in A'} H^g \text{ for all } A' \subsetneq A.$$

Define $H(G, G/H)$ = maximum size of a subset $A \subseteq G$ such that

$$\cap_{g \in A} H^g < \cap_{g \in A'} H^g \text{ for all } A' \subsetneq A.$$

Proof sketch

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\max}{<} G$.

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\text{max}}{<} G$.

Lemma $\bigcap_{x \in A} M_x < \bigcap_{x \in A'} M_x$ for all $A' \subsetneq A \subseteq S$.

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\max}{<} G$.

Lemma $\bigcap_{x \in A} M_x < \bigcap_{x \in A'} M_x$ for all $A' \subsetneq A \subseteq S$.

By [Gill & Liebeck | 2022, H | 2023], $H(G, G/M) \leqslant 177r^8 + \omega(f)$ for all $M \underset{\max}{<} G$,

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\max}{<} G$.

Lemma $\bigcap_{x \in A} M_x < \bigcap_{x \in A'} M_x$ for all $A' \subsetneq A \subseteq S$.

By [Gill & Liebeck | 2022, H | 2023], $H(G, G/M) \leqslant 177r^8 + \omega(f)$ for all $M \underset{\max}{<} G$, so only this many M_x are in a given conjugacy class.

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\max}{<} G$.

Lemma $\bigcap_{x \in A} M_x < \bigcap_{x \in A'} M_x$ for all $A' \subsetneq A \subseteq S$.

By [Gill & Liebeck | 2022, H | 2023], $H(G, G/M) \leqslant 177r^8 + \omega(f)$ for all $M \underset{\max}{<} G$, so only this many M_x are in a given conjugacy class.

We are done if G has a small number of classes of maximal subgroups.

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\max}{<} G$.

Lemma $\bigcap_{x \in A} M_x < \bigcap_{x \in A'} M_x$ for all $A' \subsetneq A \subseteq S$.

By [Gill & Liebeck | 2022, H | 2023], $H(G, G/M) \leqslant 177r^8 + \omega(f)$ for all $M \underset{\max}{<} G$, so only this many M_x are in a given conjugacy class.

We are done if G has a small number of classes of maximal subgroups.

Applying [Aschbacher | 1986] and [Liebeck & Seitz | 1990], each M_x is in one of at most $a'(r + \omega(f))^2$ classes,

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\max}{<} G$.

Lemma $\bigcap_{x \in A} M_x < \bigcap_{x \in A'} M_x$ for all $A' \subsetneq A \subseteq S$.

By [Gill & Liebeck | 2022, H | 2023], $H(G, G/M) \leqslant 177r^8 + \omega(f)$ for all $M \underset{\max}{<} G$, so only this many M_x are in a given conjugacy class.

We are done if G has a small number of classes of maximal subgroups.

Applying [Aschbacher | 1986] and [Liebeck & Seitz | 1990], each M_x is in one of at most $a'(r + \omega(f))^2$ classes, or one $\langle S \setminus x \rangle$ is an almost simple group of:

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\max}{<} G$.

Lemma $\bigcap_{x \in A} M_x < \bigcap_{x \in A'} M_x$ for all $A' \subsetneq A \subseteq S$.

By [Gill & Liebeck | 2022, H | 2023], $H(G, G/M) \leqslant 177r^8 + \omega(f)$ for all $M \underset{\max}{<} G$, so only this many M_x are in a given conjugacy class.

We are done if G has a small number of classes of maximal subgroups.

Applying [Aschbacher | 1986] and [Liebeck & Seitz | 1990], each M_x is in one of at most $a'(r + \omega(f))^2$ classes, or one $\langle S \setminus x \rangle$ is an almost simple group of:

1 small order:

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\max}{<} G$.

Lemma $\bigcap_{x \in A} M_x < \bigcap_{x \in A'} M_x$ for all $A' \subsetneq A \subseteq S$.

By [Gill & Liebeck | 2022, H | 2023], $H(G, G/M) \leqslant 177r^8 + \omega(f)$ for all $M \underset{\max}{<} G$, so only this many M_x are in a given conjugacy class.

We are done if G has a small number of classes of maximal subgroups.

Applying [Aschbacher | 1986] and [Liebeck & Seitz | 1990], each M_x is in one of at most $a'(r + \omega(f))^2$ classes, or one $\langle S \setminus x \rangle$ is an almost simple group of:

1 small order:

2 Lie type of rank $r_0 < r$ over $\mathbb{F}_{p^{f_0}}$ with $f_0 \leqslant f$:

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\max}{<} G$.

Lemma $\bigcap_{x \in A} M_x < \bigcap_{x \in A'} M_x$ for all $A' \subsetneq A \subseteq S$.

By [Gill & Liebeck | 2022, H | 2023], $H(G, G/M) \leqslant 177r^8 + \omega(f)$ for all $M \underset{\max}{<} G$, so only this many M_x are in a given conjugacy class.

We are done if G has a small number of classes of maximal subgroups.

Applying [Aschbacher | 1986] and [Liebeck & Seitz | 1990], each M_x is in one of at most $a'(r + \omega(f))^2$ classes, or one $\langle S \setminus x \rangle$ is an almost simple group of:

1. small order: bound using length of $\langle S \setminus x \rangle$
2. Lie type of rank $r_0 < r$ over $\mathbb{F}_{p^{f_0}}$ with $f_0 \leqslant f$:

Proof sketch

Let G be an almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$.

Let S be a minimal generating set. We'll show that $|S| \leqslant a(r + \omega(f))^{10}$.

For $x \in S$, fix $\langle S \setminus x \rangle \leqslant M_x \underset{\max}{<} G$.

Lemma $\bigcap_{x \in A} M_x < \bigcap_{x \in A'} M_x$ for all $A' \subsetneq A \subseteq S$.

By [Gill & Liebeck | 2022, H | 2023], $H(G, G/M) \leqslant 177r^8 + \omega(f)$ for all $M \underset{\max}{<} G$, so only this many M_x are in a given conjugacy class.

We are done if G has a small number of classes of maximal subgroups.

Applying [Aschbacher | 1986] and [Liebeck & Seitz | 1990], each M_x is in one of at most $a'(r + \omega(f))^2$ classes, or one $\langle S \setminus x \rangle$ is an almost simple group of:

1. small order: bound using length of $\langle S \setminus x \rangle$
2. Lie type of rank $r_0 < r$ over $\mathbb{F}_{p^{f_0}}$ with $f_0 \leqslant f$: apply induction on r.

How big can a minimal generating set be?

How big can a minimal generating set be?

Theorem (H | 2023)

If G is finite, then $m(G) \leqslant a(\sum_{p \text{ prime}} d(G_p))^b$. [e.g. $a = 10^{10}$ & $b = 10$]

How big can a minimal generating set be?

Theorem (H | 2023)

If G is finite, then $m(G) \leqslant a(\sum_{p \text{ prime}} d(G_p))^b$. [e.g. $a = 10^{10}$ & $b = 10$]

$\uparrow$

Theorem (H | 2023)

If G is a finite almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$, for prime p, then $m(G) \leqslant a(r + \omega(f))^b$. [e.g. $a = 10^5$ & $b = 10$]

How big can a minimal generating set be?

Theorem (H | 2023)

If G is finite, then $m(G) \leqslant a(\sum_{p \text{ prime}} d(G_p))^b$. [e.g. $a = 10^{10}$ & $b = 10$]

$\uparrow$

Theorem (H | 2023)

If G is a finite almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$, for prime p, then $m(G) \leqslant a(r + \omega(f))^b$. [e.g. $a = 10^5$ & $b = 10$]

$\uparrow$

Theorem (Gill & Liebeck | 2022, H | 2023)

If G is a finite almost simple group of Lie type of rank r over $\mathbb{F}_{p^f}$, for prime p, acting primitively on X, then $B(G, X) \leqslant ar^b + \omega(f)$. [e.g. $a = 177$ & $b = 8$]