Is a finite group ever the union of conjugates of two equal-sized proper subgroups?

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Topics in Group Theory

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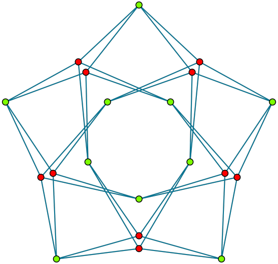
10 September 2024



Joint work with

David Ellis

(University of Bristol)

arXiv > math > arXiv:2408.16064

Mathematics > Group Theory

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Orbits of permutation groups with no derangements

David Ellis, Scott Harper

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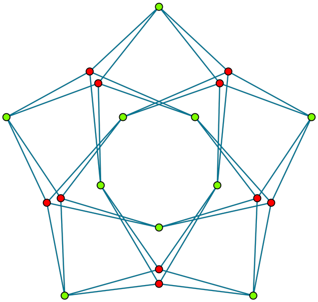

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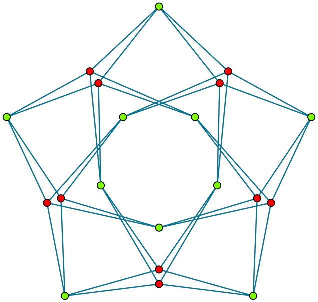

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True when Γ is 3- or 4-regular.

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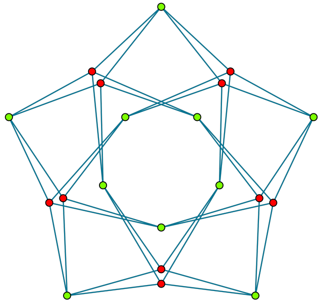

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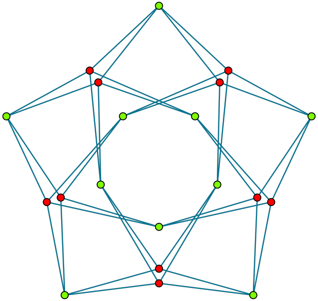

4 Extremal combinatorics

Erdős–Ko–Rado theorem for intersecting families of permutations.

[Nakajima | 2022]

Proof Ideas

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Main Theorem [Ellis & H | 2024]

Let $1 \neq G \leqslant \mathrm{Sym}(n)$ with orbits Ω_1 and Ω_2. Assume $|\Omega_1| = |\Omega_2|$. Assume G is primitive on Ω_2. Then G has a derangement.

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If $G \leqslant \mathrm{Sym}(\Omega_1)$ is almost simple, then use [Bubbloni, Spiga & Weigel | 2024].

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Assume $\overline{H} \leqslant \mathrm{Sym}(\Omega)$ is ~~almost~~ simple ~~or product action~~, as otherwise $\overline{H}$ has a derangement of order p. [Burness, Giudici & Wilson | 2011]

Have projective rep $\lambda \colon H/Z(H) \to \mathrm{PGL}_d(p)$. Assume absolutely irreducible.

Simplifying assumption λ is the lift of a projective rep $\overline{H} \to \mathrm{PGL}_d(p)$.

Example $\overline{H} = \mathrm{PSL}_n(p)$ $\quad |\overline{H}| = p^{\frac{1}{2}n(n-1)}(p^2-1)\cdots(p^d-1)$

Note p^d divides $|\Omega| = |H : H_\omega|$ which properly divides $|\overline{H}|$, so $d < \frac{1}{2}n(n-1)$.

Then $d = n$ and λ is the lift of the natural rep (or its dual). [Liebeck | 1985]

Argue directly, e.g. if $H_\omega \approx \mathrm{SL}_{n/2}(p^2)$, choose h with odd-dim fixed space.

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Key to this are results of [H & Liebeck | 2024] extending [Feit & Tits | 1978].