Generating infinite simple groups

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on work joint with Collin Bleak, Casey Donoven & James Hyde

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Thompson's group V was the first example of an infinite finitely presented simple group, and for decades all known examples were related to V.

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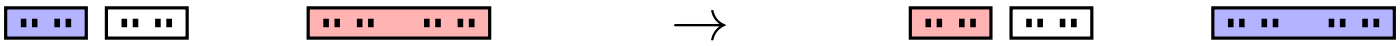

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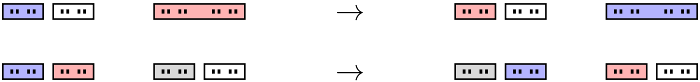

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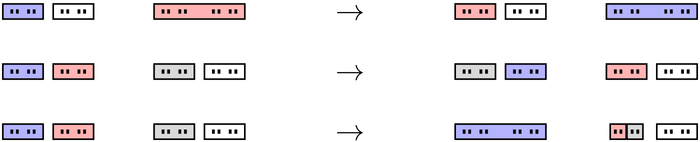

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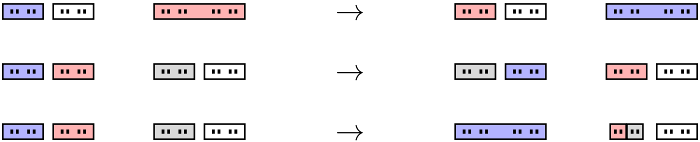

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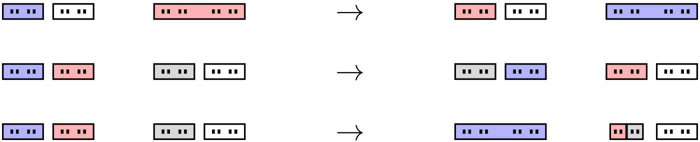

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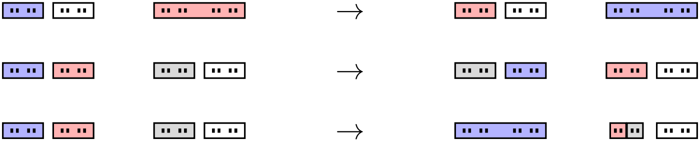

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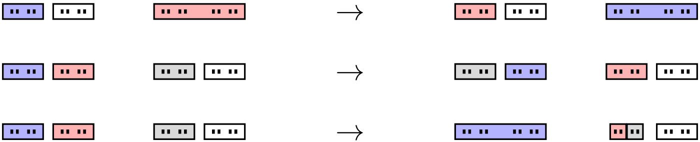

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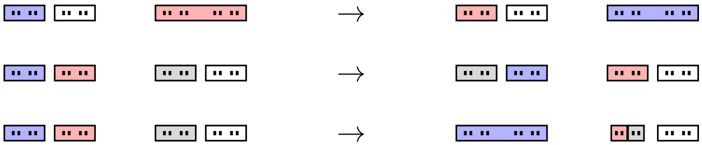

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This is the first nontrivial example of an infinite group with this property.

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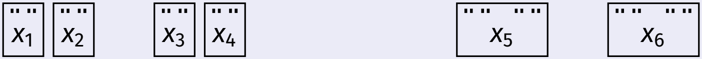

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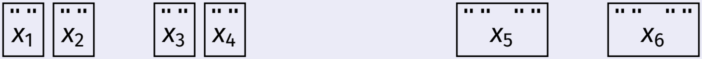

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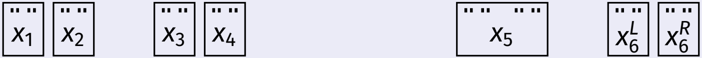

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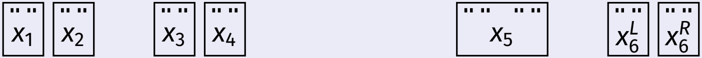

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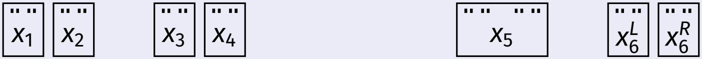

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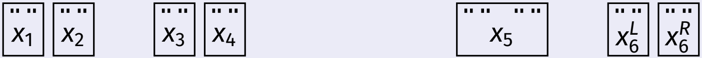

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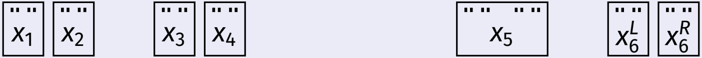

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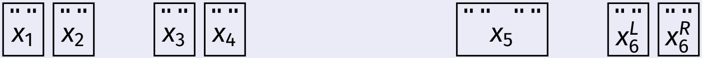

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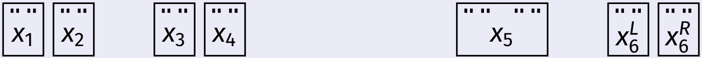

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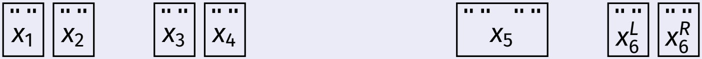

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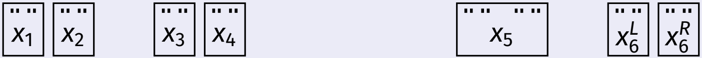

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$$a = (x_1\ x_2)(x_3\ x_4)(x_5\ x_6) \quad b = (x_2\ x_3)(x_5\ x_6) \quad c = (x_1\ x_5)(x_2\ x_6^L)(x_3\ x_6^R).$$

To prove this, we apply the generation criterion:

$$\langle ab, c \rangle = \langle (x_1\ x_3\ x_4\ x_2), (x_1\ x_5)(x_2\ x_6^L)(x_3\ x_6^R) \rangle = \mathrm{Sym}(X')$$
$$\langle a, b, c \rangle \geqslant \langle (x_1\ x_2\ x_3\ x_4\ x_5), b(x_2\ x_3) \rangle = \mathrm{Sym}(X).$$

That V and some generalisations are generated by three involutions was recently and independently proved by [Schesler, Skipper, Wu | 2024$^{\text{arxiv}}$].

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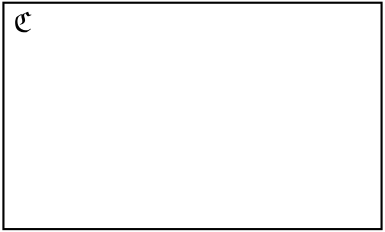

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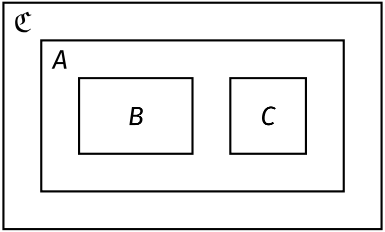

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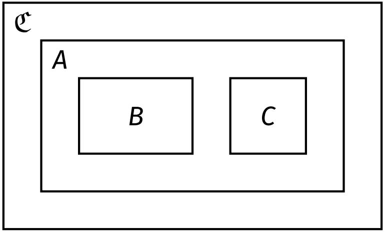

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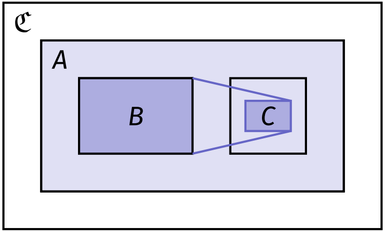

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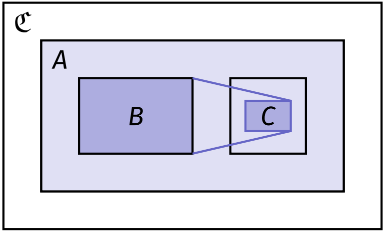

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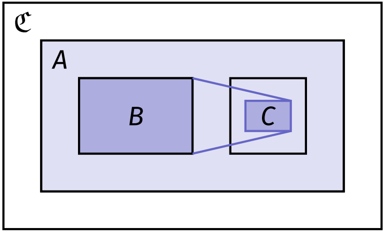

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Vigorous groups include all generalisations of V and most of the other known finitely presented simple groups (e.g. Higman–Thompson groups V_n, Brin–Thompson groups nV and Nekrashevych's dynamical groups).

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The following special case of **2** answers a question of [Sapir | 2017].

Corollary We have $V = \langle x, y \rangle$ where $|x| = 2$ and $|y| = 3$.

(2, 3)-generation of simple vigorous groups

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fin. gen. simple
vigorous group G

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$\mathfrak{C}$

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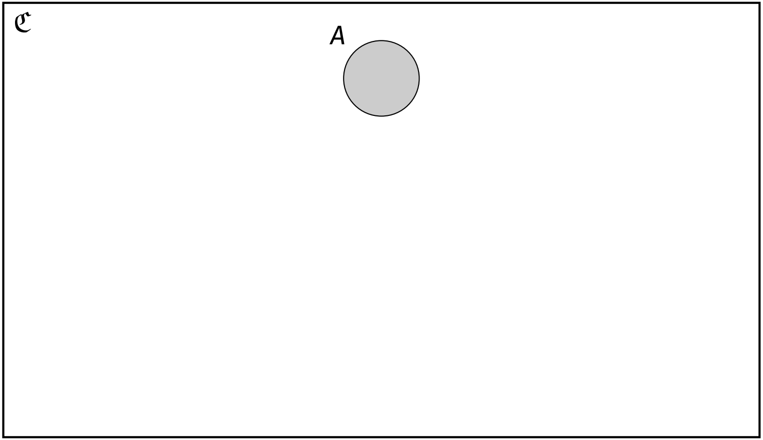

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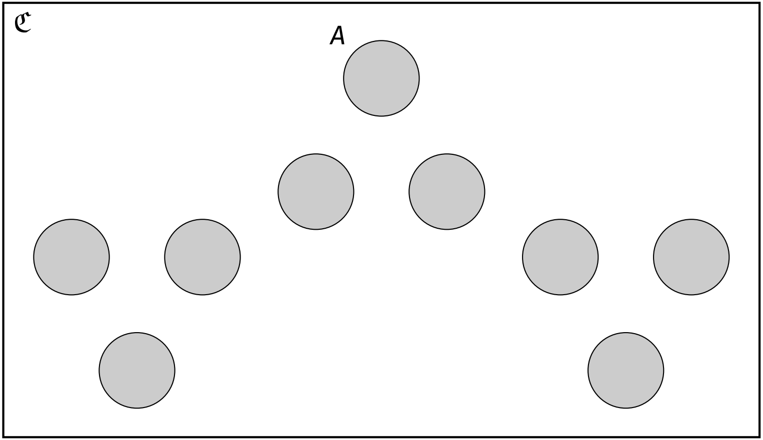

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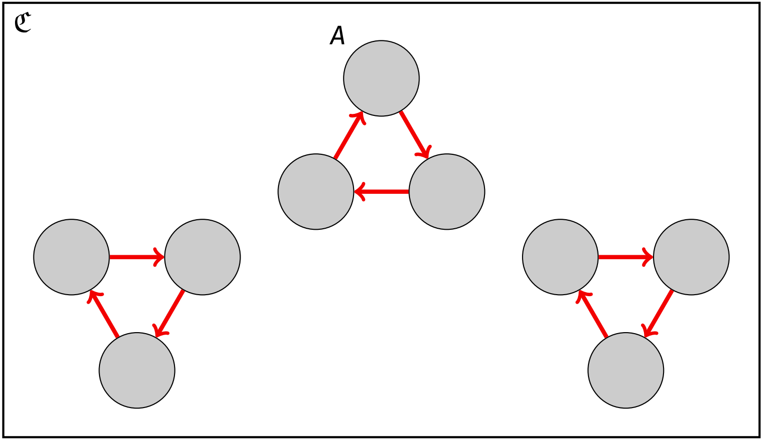

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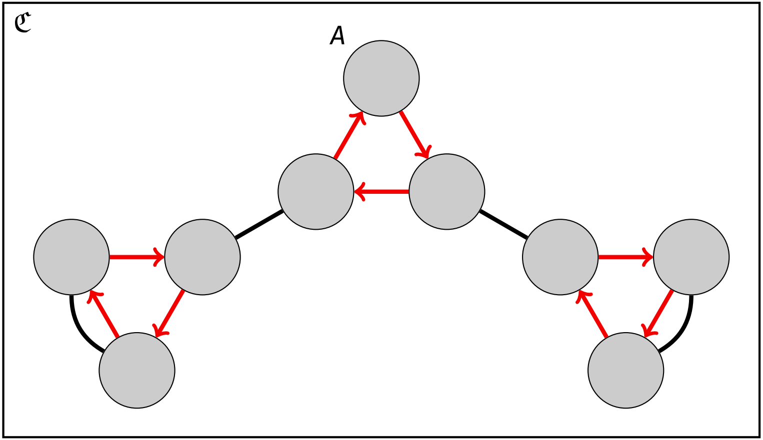

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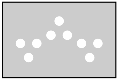

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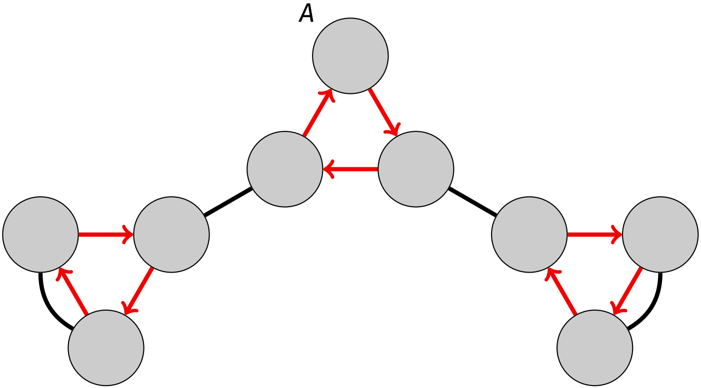

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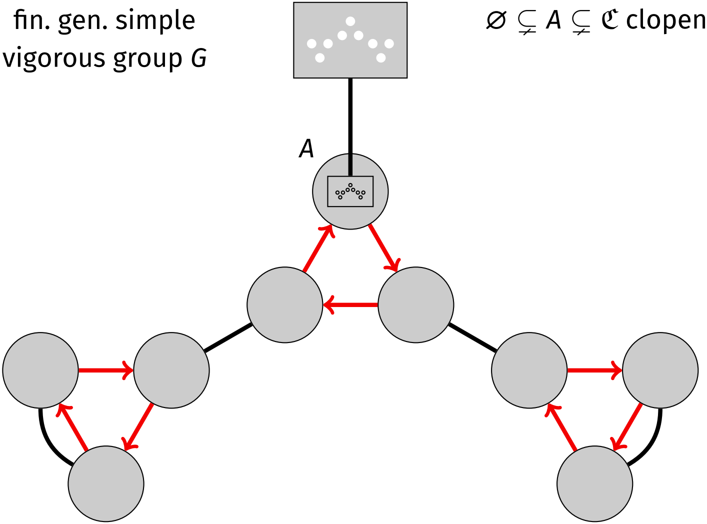

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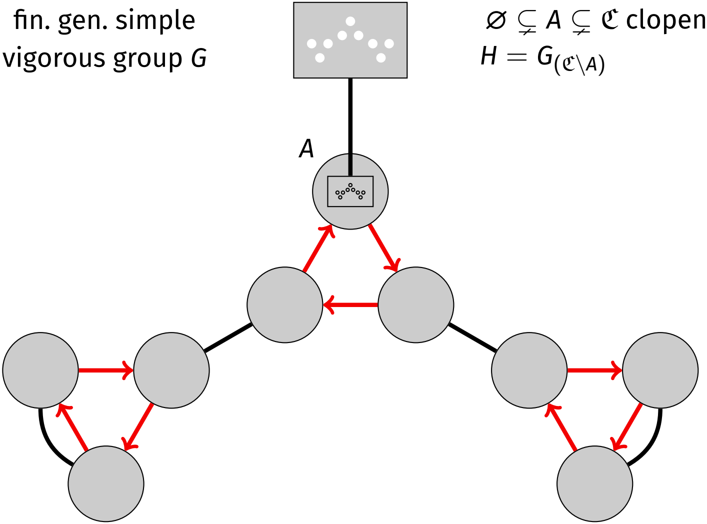

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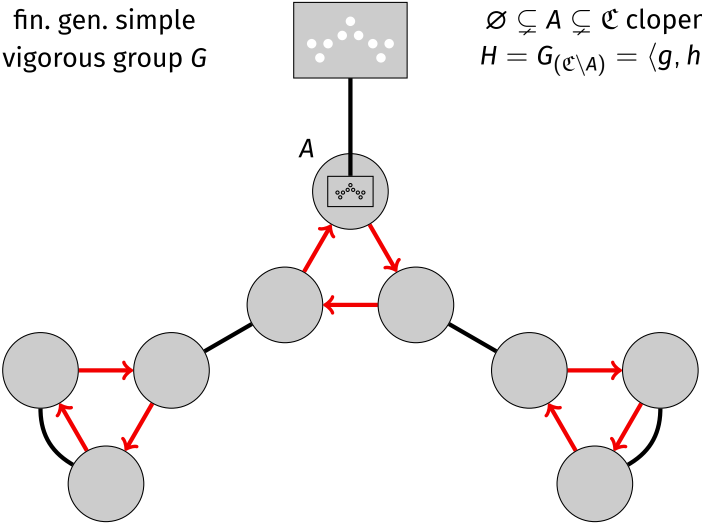

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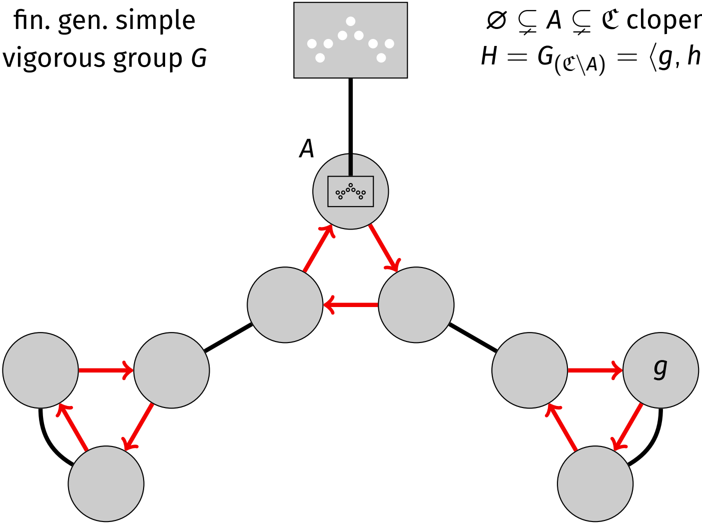

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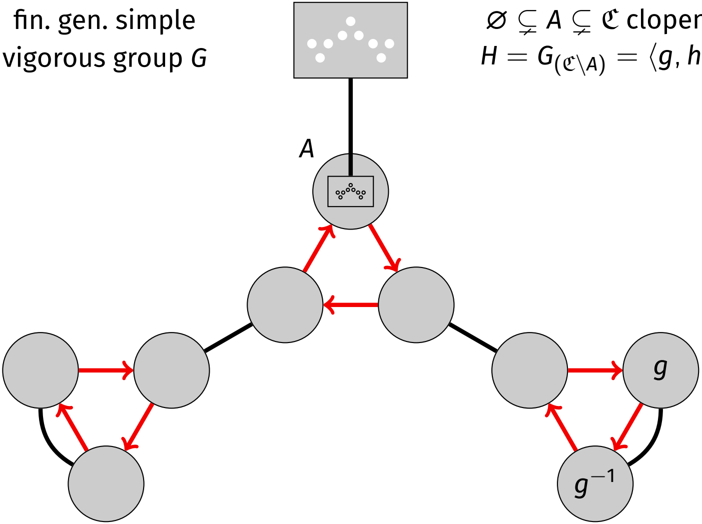

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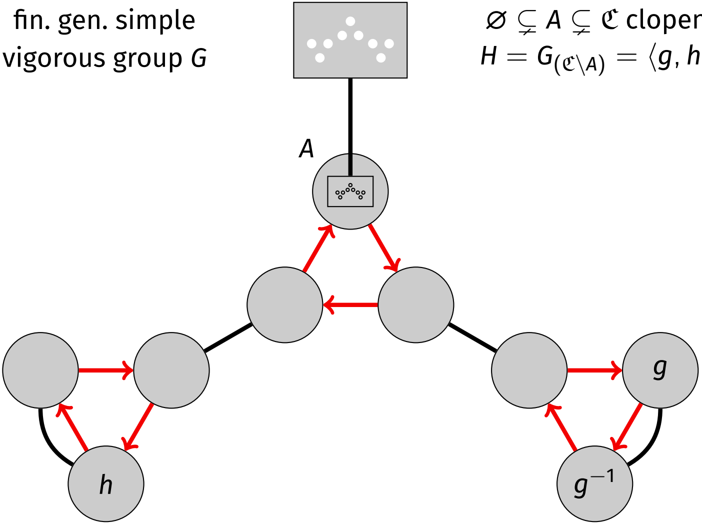

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