# Topics in Discrete Mathematics

## Problem Sheet 1

**There is no assessed homework on this sheet.**

If you wish, you may hand in solutions to **Problems 3, 5, 7, 9** for marking.

**Latin squares**

1. Arrange the four Jacks, Queens, Kings and Aces from a standard pack of playing cards in a four by four grid such that no rank or suit is repeated in any row or column.

2. Complete the following array to make a Latin square

| | | | |
|---|---|---|---|
| | | | 4 |
| | 4 | 3 | |
| | | 2 | |
| 2 | 1 | 4 | |

3. Show that the number of Latin squares of order $n$, with entries in $\{1, 2, \ldots, n\}$, is 1, 2, 12 when $n$ is 1, 2, 3, respectively.

4. Prove that there do not exist two orthogonal Latin squares of order 2.

5. Use the field with four elements to write down 3 mutually orthogonal Latin squares of order 4.

6. Write down the direct product $A \times B$ where

$$A = \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ \hline \end{array} \quad \text{and} \quad B = \begin{array}{|ccc|} \hline 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ \hline \end{array}$$

7. Write down two orthogonal Latin squares of order 12.

8. Let $A = (a_{ij})$ be a Latin square of order $n$, whose rows, columns and symbols are indexed by $S = \{1, 2, \ldots, n\}$. Let $X = \{(i, j, k) \in S^3 \mid a_{ij} = k\}$.
   (a) Define $B = (b_{ij})$ as $b_{ij} = k$ if $(j, i, k) \in X$. Prove that $B$ is a Latin square.
   (b) Define $C = (c_{ij})$ as $c_{ij} = k$ if $(i, k, j) \in X$. Prove that $C$ is a Latin square.

9. Let $A = (a_{ij})$ and $B = (b_{ij})$ be Latin squares of order $n$. Prove that $A$ and $B$ are orthogonal if and only if for every $(x, y) \in \{1, 2, \ldots, n\}^2$ there exists $(i, j) \in \{1, 2, \ldots, n\}^2$ such that $(a_{ij}, b_{ij}) = (x, y)$.

10. Prove that there exist $k + 2$ mutually orthogonal $n \times n$ arrays if and only if there exist $k$ mutually orthogonal Latin squares of order $n$.

**Optional problems for students who are familiar with groups**

1*. (a) Prove that the Cayley table of a finite group is a Latin square.

(b) Write down a Latin square that is not the Cayley table of a group. Justify your example.

2*. Let $A = (a_{ij})$ and $B = (b_{ij})$ be Latin squares of order $n$. Then $A$ is *isotopic* to $B$ if $a_{ij} = \rho(b_{\sigma(i)\tau(j)})$ for some permutations $\rho, \sigma, \tau$ on $n$ points. Write $A \sim B$ if $A$ is isotopic to $B$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Assume $A \sim B$. Prove that $A$ has an orthogonal mate if and only if $B$ does.

3*. Let $A = (a_{ij})$ be a Latin square of order $n$. Then $A$ satisfies the *quadrangle condition* if for all $i_1, i_2, j_1, j_2, k_1, k_2, l_1, l_2 \in \{1, \ldots, n\}$ we have

$$(a_{i_1 j_1} = a_{k_1 l_1} \ \& \ a_{i_1 j_2} = a_{k_1 l_2} \ \& \ a_{i_2 j_1} = a_{k_2 l_1}) \implies a_{i_2 j_2} = a_{k_2 l_2}$$

(a) Let $A$ be the Cayley table of a finite group. Prove that $A$ satisfies the quadrangle condition.

(b) Assume that $A \sim B$. Prove that $A$ satisfies the quadrangle condition if and only if $B$ does.

(c) Prove that $A$ satisfies the quadrangle condition if and only if $A$ is isotopic to the Cayley table of a finite group.