Topics in Discrete Mathematics

Problem Sheet 2

Assessed Homework 1

1. For each $n \times n$ array with entries in $\{0, 1, \ldots, n-1\}$, determine, with proof, whether it is a Latin square, and if it is, determine, with proof, whether it has an orthogonal mate:
   (a) the 7×7 array $A = (a_{ij})$ defined as $a_{ij} = i + 2j \pmod 7$
   (b) the 8×8 array $B = (b_{ij})$ defined as $b_{ij} = i - j \pmod 8$
   (c) the 9×9 array $C = (c_{ij})$ defined as $c_{ij} = i + 3j \pmod 9$.
2. Referring to the axioms defining a projective plane, write down (without proof) an example of a set $\mathcal{P}$ and a set $\mathcal{L}$ of subsets of $\mathcal{P}$ that satisfy
   (a) Axioms (1) and (2) but not Axiom (3)
   (b) Axioms (1) and (3) but not Axiom (2)
   (c) Axioms (2) and (3) but not Axiom (1).

Latin squares

3. (a) Write down a Latin square A with symbols $1, 2, 3, 4$ that has an orthogonal mate.
   (b) Write down a Latin square B with symbols $1, 2, 3, 4$ that does not have an orthogonal mate.
   (c) Show that any Latin square of order four, with symbols $1, 2, 3, 4$, can be obtained from either A or B by permuting the rows, columns and symbols.
4. Let $A = (a_{ij})$ be the Latin square with entries in $\{0, 1, \ldots, n-1\}$ defined by $a_{ij} = i + j \pmod n$. Prove that A has an orthogonal mate if and only if n is odd.
5. Let A be a Latin square of order n. Assume that A is the union of disjoint transversals $T_1, T_2, \ldots, T_n$. Define $B = (b_{ij})$ as $b_{ij} = k$ if and only if $(i, j) \in T_k$. Prove that B is a Latin square orthogonal to A.
6. Write down a Latin square of order 7 that has no orthogonal mate.

Projective planes

7. (a) What is the order of a projective plane such that each point is on exactly 8 lines?
   (b) How many lines does a projective plane of order 8 have?
   (c) Does there exist a projective plane with exactly 8 lines?
8. Let $(\mathcal{P}, \mathcal{L})$ be a projective plane of order n. A *triangle* is a set of three distinct points of $\mathcal{P}$ that are not contained in a common line. Prove that the number of triangles in $(\mathcal{P}, \mathcal{L})$ is
$$n^2(n^2 + n)(n^2 + n + 1)/6.$$
9. Let $(\mathcal{P}, \mathcal{L})$ be a projective plane. Let $p \in \mathcal{P}$. Prove, directly from the defining axioms of a projective plane, that there exists $l \in \mathcal{L}$ such that $p \notin l$.

The following problem is quite involved.

10. Let $\mathcal{P}$ be a finite set and let $\mathcal{L}$ be a set of subsets of $\mathcal{P}$. Assume that there exists $n \geqslant 2$ such that

    (I) $|\mathcal{P}| = n^2 + n + 1$

    (II) $|l| = n + 1$ for all $l \in \mathcal{L}$

    (III) for any distinct $p, p' \in \mathcal{P}$ there exists a unique $l \in \mathcal{L}$ such that $p, p' \in l$.

    The aim of this problem is to prove that $(\mathcal{P}, \mathcal{L})$ is a finite projective plane of order n.

    (a) Prove that $(\mathcal{P}, \mathcal{L})$ satisfies Axiom (1).

    (b) (i) Prove that $|\mathcal{L}| = n^2 + n + 1$.

    (ii) Prove that any $p \in \mathcal{P}$ is contained in exactly $n + 1$ subsets $l \in \mathcal{L}$.

    (iii) Hence, or otherwise, prove that $(\mathcal{P}, \mathcal{L})$ satisfies Axiom (2).

    (c) Prove that $(\mathcal{P}, \mathcal{L})$ satisfies Axiom (3).