Topics in Discrete Mathematics

Problem Sheet 3

Assessed Homework 2

1. Let N be the smallest positive integer strictly greater than 1 for which there exist 41 mutually orthogonal Latin squares of order N. Determine
   (a) the value of N
   (b) the maximal size of a set of mutually orthogonal Latin squares of order N.

   Justify your answers. You may use, without proof, any result from the lecture notes.
2. Write $X = \mathbb{Z}/v\mathbb{Z}$. Let $S \subseteq X$ be a (v, k, λ) cyclic difference set. Prove that the complement $X \setminus S$ is a $(v, v-k, v-2k+\lambda)$ cyclic difference set.

Projective planes

3. Does there exist a finite projective plane of order 21?
4. (a) Draw $\mathrm{P}_2(\mathbb{F}_3)$.
   (b) From $\mathrm{P}_2(\mathbb{F}_3)$, construct two orthogonal Latin squares of order 3.
5. Let $l_\infty = \{p_a \mid a \in \mathbb{Z} \cup \{\infty\}\}$. Define $\mathcal{P} = \mathbb{Z}^2 \cup l_\infty$ and $\mathcal{L} = \{l_\infty\} \cup \bigcup_{a \in \mathbb{Z} \cup \{\infty\}} \mathcal{L}_a$, where

$$\mathcal{L}_a = \begin{cases} \{\{(x, ax+b) \mid x \in \mathbb{Z}\} \cup \{p_a\}\} \mid b \in \mathbb{Z}\} & \text{if } a \in \mathbb{Z} \\ \{\{(c, y) \mid y \in \mathbb{Z}\} \cup \{p_\infty\}\} \mid c \in \mathbb{Z}\} & \text{if } a = \infty. \end{cases}$$

   Is $(\mathcal{P}, \mathcal{L})$ a projective plane?
6. (a) Let $\mathcal{P}$ be a set and let $\mathcal{L}$ be a set of subsets of $\mathcal{P}$. The *dual* of $(\mathcal{P}, \mathcal{L})$ is $(\mathcal{P}', \mathcal{L}')$ where $\mathcal{P}' = \mathcal{L}$ and $\mathcal{L}' = \{\{l \in \mathcal{L} \mid p \in l\} \mid p \in \mathcal{P}\}$. Is the dual of a projective plane a projective plane?
   (b) Projective planes $(\mathcal{P}_1, \mathcal{L}_1)$ and $(\mathcal{P}_2, \mathcal{L}_2)$ are *equivalent* if is a bijection $f\colon \mathcal{P}_1 \to \mathcal{P}_2$ such that for $l \subseteq \mathcal{P}_1$ we have $l \in \mathcal{L}_1$ if and only if $f(l) \in \mathcal{L}_2$. Is the Fano plane equivalent to its dual?

Cyclic difference sets

7. Let $S \subseteq \mathbb{Z}/40\mathbb{Z}$ be a $(40, k, 4)$ cyclic difference set. Determine k.
8. Verify directly Proposition 14.4 with $m = 3$.
9. Write down
   (a) a $(23, 11, 5)$ cyclic difference set
   (b) a $(19, 10, 5)$ cyclic difference set
   (c) a $(11, 5, 2)$ cyclic difference set that contains 2.
10. Assume that v and k are coprime. Let $S \subseteq \mathbb{Z}/v\mathbb{Z}$ be a (v, k, λ) cyclic difference set. Prove that there exists a unique $a \in \mathbb{Z}/v\mathbb{Z}$ such that the sum of elements of $S + a$ is zero.